Everything you ever wanted to know about the thermodynamics of 2-D Black Holes.

(And some black holes in higher dimensions, as well.)

Based on hep-th/0703230, w. Daniel Grumiller
and hep-th/0411121 w. Josh Davis

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Syracuse University, April 2007

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Outline

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- Sacrifice integrability. Assume coupled to unspecified, dynamical matter.
- Attempt a rigorous semiclassical analysis.

The Semiclassical Limit

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- Ω is the thermodynamic potential for the appropriate ensemble, β is the periodicity of the Euclidean time.

Saddle-point Approximation

Consider a small perturbation around a classical solution

$$I_E[g_{cl} + \delta g, X_{cl} + \delta X] = I_E[g_{cl}, X_{cl}] + \delta I_E[g_{cl}, X_{cl}; \delta g, \delta X] + \frac{1}{2}\, \delta^2 I_E[g_{cl}, X_{cl}; \delta g, \delta X] + \dots$$

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$$\mathcal{Z} \sim \exp\left(-\frac{1}{\hbar}\, I_E[g_{cl}, X_{cl}]\right) \int \mathscr{D}\delta g\, \mathscr{D}\delta X \exp\left(-\frac{1}{2\hbar}\,\delta^2 I_E\right) \times \ldots$$

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The semiclassical analysis is much more involved than we might have guessed!

Outline

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The standard form of the action for this theory is

$$I_E = -\frac{1}{16\pi G_2}\int_{\mathcal{M}} d^2x\sqrt{g}\,\left[X\,R - U(X)\,(\nabla X)^2 - 2\,V(X)\right]$$
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- The boundary integral is the dilaton gravity analog of the Gibbons-Hawking-York boundary term.

The Equations of Motion

Extremize the action: $\delta I_E = 0$

$$U(X)\,\nabla_\mu X \nabla_\nu X - \frac{1}{2}\,g_{\mu\nu} U(X) (\nabla X)^2 - g_{\mu\nu} V(X) + \nabla_\mu \nabla_\nu X - g_{\mu\nu} \nabla^2 X = 0$$

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All results are independent of this choice of gauge. Our analysis was performed using the general form of the metric.

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We will frequently use this function.

Black Holes

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- $\xi(X) \sim e^Q\, w$ as $X \to \infty$.

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- The horizon occurs at $w(X_h) = 2M$, where $\xi(X_h) = 0$.
- If $\exists$ multiple X_h then we always take the outermost horizon.

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- $\xi^{-1/2}$ is the 'Tolman factor'.

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Need a limiting procedure to calculate the action. Implement this in a coordinate-independent way by putting a regulator on the dilaton.

$$\boxed{X \leq X_{\text{reg}}}$$

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This last point is especially important. We also find

$$\lim_{X_{\text{reg}} \to \infty} \delta I_E^{\text{reg}} \neq 0$$

This means the semiclassical limit isn't well defined.

Variational Properties of the Action

Consider small, independent variations of $g_{\mu\nu}$ and X

$$\delta I_E = \int_{\mathcal{M}} d^2x \sqrt{g}\, \left[\mathcal{E}^{\mu\nu} \delta g_{\mu\nu} + \mathcal{E}_X \delta X \right] + \int_{\partial\mathcal{M}} dx \sqrt{\gamma}\, \left[\pi^{ab} \delta\gamma_{ab} + \pi_X \delta X \right]$$

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$$\delta I_E^{\text{reg}} = \int d\tau \left[-\frac{1}{2}\,\partial_r X\, \delta\xi + \ldots\right]$$

This needs to vanish for all $\delta\xi$ that preserve the boundary conditions on the fields.

What $\delta\xi$ preserve the boundary conditions?

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What does this mean for δI? Using $\partial_r X = e^{-Q}$, we get

$$\delta I = \int d\tau \delta M \neq 0$$

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- Determine I_{CT} so that Γ is finite, $\delta\Gamma = 0$ on-shell.

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H-J equation reduces to a linear diff-eq. Easy to Solve:

$$I_{CT} = - \int_{\partial\mathcal{M}} dx \sqrt{\gamma} \, \sqrt{w(X) \, e^{-Q(X)}}$$

The Improved Action

The correct action for 2-D dilaton gravity is

$$\Gamma = -\,\frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\,\left[X R - U(X)\,(\nabla X)^2 - 2\,V(X)\right] - \int_{\partial\mathcal{M}} dx\sqrt{\gamma}\,X\,K + \int_{\partial\mathcal{M}} dx\sqrt{\gamma}\,\sqrt{w(X)\,e^{-Q(X)}}$$

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$$\boxed{\Gamma\big|_{\text{eom}} = \beta\,(M - 2\pi X_h\,T)}$$

3. First variation $\delta\Gamma$ vanishes on-shell $\forall\ \delta g_{\mu\nu}$ and δX that preserve the boundary conditions.

$$\boxed{\delta\Gamma\big|_{\text{eom}} = 0}$$

The Euclidean Path Integral

A sensible starting point. Recovers 'classical' physics as $\hbar \to 0$.

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Well-defined saddle point approximation?

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- Linear term vanishes.

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- Leading term is finite.
- Linear term vanishes.
- Still have to worry about the quadratic term.

Outline

A Black Hole in a Box

Follow the approach due to York (PRD **33**, 1986). Consider the Schwarzschild BH in 4-D.

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- Properly accounts for the non-linear effects of gravitational binding energy.
- Incoporates the dilaton charge and its chemical potential.

$$\lim_{X_c \to \infty} \psi_c\, dD_c = 0$$

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2. Then the specific heat simplifies to

$$C_D = \frac{\epsilon}{T} + \mathcal{O}(\epsilon^2)$$

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So we obtain (Gibbons & Perry '92)

$$T = \frac{1}{\pi\,\sqrt{\alpha'}} \quad S = 2\pi X_h \quad M = T\,S$$

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We can use X_c in our calculations, but we have to treat it like the regulator we used earlier. Must take $X_c \to \infty$ limit in all calculations!

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Geometry encodes some info about worldsheet CFT: $k \geq 2$

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$$F = -b\sqrt{1-\frac{2}{k}}\operatorname{arcsinh}\sqrt{k(k-2)}$$

Manifestly non-positive. Stable against tunneling to 'CDV'.

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Conclude that, in 2-D, string theory is its own reservoir. It is self-contained and self-consistent.

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Consider gravity in $d+1$ dimensions:

$$I_{d+1} = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^{\,d+1}x\sqrt{g}\,(R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^{\,d}x\sqrt{\gamma}(K+\ldots)$$

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Solutions with a $d-1$ sphere

$$ds^2 = \xi(r)d\tau^2 + \frac{1}{\xi(r)}\,dr^2 + G^{\frac{2}{d-1}}\varphi(r)^2 d\Omega_{d-1}^2$$

Reduce on-shell action on S^{d-1}. Looks like 2-D DG action. Our thermo results apply!

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Caveat: Can't 'lift' the 2-D counterterm to higher dimensions.

The S-wave Reduction

The s-wave reduction of the on-shell action looks like a 2-D DG model with

$$X = \Upsilon\, G\, \varphi(r)^{d-1} \qquad \Upsilon := \frac{A_{d-1}}{8\pi G}$$

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Cosmological constant:

$$\Lambda := e\,\frac{d(d-1)}{2\,\ell^2} \qquad e = \pm 1, 0$$

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Now apply all the results of our thermo analysis.

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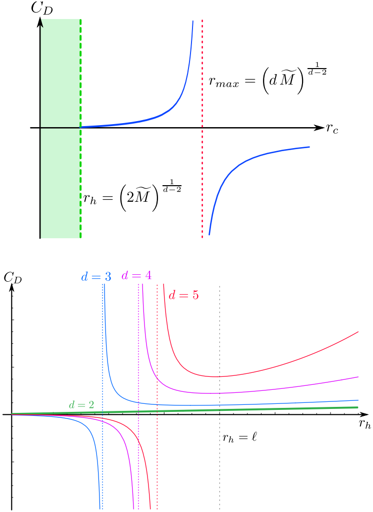
3. The internal energy is related to $Q_{\partial_\tau} = M$ by

$$\lim_{r_c \to \infty} \sqrt{\xi_c} E_c = M$$

Thermodynamic Stability

The sign of the specific heat depends on Λ, M, X_c.



Outline

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Next Stage

- Include additional degree of freedom
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- Calculate tunneling rates in Cosmology?