7 $\text{AdS}_3/\text{CFT}_2$

The discovery of AdS/CFT by Maldacena in the end of 1997 had a profound impact on physics. In this last chapter we discuss in some detail aspects of the $\text{AdS}_3/\text{CFT}_2$ correspondence.

We start with a discussion of general aspects of AdS/CFT before summarizing the $\text{AdS}_3/\text{CFT}_2$ results that we have obtained already in these lecture. Then we consider four additional checks of the correspondence: we reconsider the Cardy formula for the entropy and its leading semi-classical correction, compare the linearized spectra around the global vacuum, calculate all stress tensor correlation functions and finally consider entanglement entropy. In all these cases we perform calculations both on the gravity side and the field theory side. The main conclusion is that all observables considered in this chapter agree precisely on both sides of the correspondence, which provides evidence for its correctness. In the final section we provide an outlook to open questions and further developments, some of which will be addressed in the second part of this lecture series.

7.1 General aspects of AdS/CFT correspondence

The main claim of AdS/CFT is that **quantum gravity in AdS is equivalent to a CFT in one dimension less, thus concretely realizing the holographic principle**. This means that for every observable on the gravity side there is a corresponding observable on the field theory side and vice versa. Therefore, there must be a dictionary between all such observables. AdS/CFT was discovered through string theory considerations, involving D-branes. See the MAGOO review for details. In its weakest form, AdS/CFT relates weakly coupled Einstein gravity (for small values of Newton's constant) to strongly coupled CFTs (for a large number of degrees of freedom, e.g. a large number of "colours" in $D = 4$ $SU(N)$ super-Yang–Mills). We are mostly concerned with checks in this (super-) gravity approximation.

The main class of observables in a field theory are correlation functions of gauge invariant operators. Thus, the most important entry in the AdS/CFT dictionary is how to relate such correlation functions with corresponding observables on the gravity side. The conjectured relation by Gubser–Klebanov–Polaykov and Witten is

$$\boxed{\Big\langle \exp\Big(\int j(x)\,\mathcal{O}(x)\Big)\Big\rangle_{\text{CFT}} = Z_{\text{gravity}}\Big[\phi(x,z)|_{z\to 0} = j(x)\Big]\,.} \tag{1}$$

The left hand side is evaluated in the CFT. Here $\mathcal{O}(x)$ is some gauge invariant operator whose source is given by $j(x)$. The expression on the left hand side is nothing but the generating functional of correlation functions, and you get arbitrary n-point functions by taking n functional derivatives with respect to the sources and then setting the sources to zero. If you know quantum field theory all these statements must be familiar to you; if not, you should acquire this knowledge by studying quantum field theory, which has applications all over physics and beyond.

The right hand side is evaluated in quantum gravity; in the super-gravity approximation this reduces to an evaluation in classical gravity. In that limit the quantity Z_{gravity} is the classical partition function evaluated with boundary conditions for the field ϕ given by the function $j(x)$ (which coincides with the source on the CFT side), where the limit $z \to 0$ denotes approaching the asymptotically AdS boundary (while x are the boundary coordinates). The field ϕ must be the one corresponding to the gauge invariant operator $\mathcal{O}$.

The claim of AdS/CFT is that for each operator on the CFT side there is a corresponding field on the gravity side and vice versa, so that the equality (1) holds.

For instance, say you want to calculate a 2-point function. Then the GKPW-prescription (1) yields

$$\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle_{\text{CFT}} = \frac{\delta^2}{\delta j(x_1)\,\delta j(x_2)}\,\Big\langle \exp\Big(\int j(x)\,\mathcal{O}(x)\Big)\Big\rangle_{\text{CFT}}\Big|_{j=0}$$
$$= \frac{\delta^2}{\delta j(x_1)\,\delta j(x_2)}\,Z_{\text{gravity}}\Big[\phi(x,z)|_{z\to 0} = j(x)\Big]\Big|_{j=0} \qquad (2)$$

where you have to make sure to take the field ϕ on the gravity side corresponding to the operator $\mathcal{O}$ on the CFT side.

Usually we are interested in the connected part of n-point functions (since the disconnected part consists of lower-point functions that we can calculate separately). The corresponding generating functional is just the logarithm of the full generating functional Z. In the supergravity approximation the gravity partition function is essentially the exponential of the (holographically renormalized) on-shell action, $Z \sim \exp(-\Gamma)$ (the minus sign is correct for Euclidean signature; in Minkowski signature we have a factor i instead; we do not keep track of this inessential factor). Taking the logarithm of Z thus yields the on-shell action Γ. In other words, we can calculate connected n-point functions in specific CFTs by calculating corresponding n functional derivatives of the action evaluated on-shell. This is one of the reasons why AdS/CFT has so many applications, as it allows to map potentially complicated calculations (n-point functions in a strongly coupled CFT) to potentially elementary calculations (taking n functional derivatives of the gravitational action evaluated on-shell).

Which operators exist in a given CFT depends very much on the details of the CFT. However, all of them have at least one operator, namely the stress energy tensor. The natural guess for the field on the gravity side corresponding to the CFT stress tensor is the metric, since it also exists universally in any (reasonable) theory of gravity. Thus, for n-point correlation functions of the CFT stress tensor the GKPW prescription in the super-gravity approximation reads

$$\langle T_{\mu_1\nu_1}(x_1)T_{\mu_2\nu_2}(x_2)\dots T_{\mu_n\nu_n}(x_n)\rangle^{\text{connected}}_{\text{CFT}}$$
$$= \frac{\delta^n}{\delta\gamma^{(0)}_{\mu_1\nu_1}(x_1)\,\delta\gamma^{(0)}_{\mu_2\nu_2}(x_2)\,\dots\delta\gamma^{(0)}_{\mu_n\nu_n}(x_n)}\,\Gamma_{\text{gravity}}\Big[g_{\mu\nu}(x,z)|_{z\to 0} = \gamma^{(0)}_{\mu\nu}(x)\Big]\Big|_{\text{EOM}} \qquad (3)$$

where Γ_{gravity} is the classical gravity action and the subscript EOM means going on-shell, which is equivalent to switching off the sources.

Thus, we have the somewhat surprising claim that, say, the 42^{nd} functional derivative of the Einstein–Hilbert AdS action with respect to the metric should reproduce the 42-point correlation function of the stress tensor in a CFT. We will check that this is actually true in an $\text{AdS}_3/\text{CFT}_2$ context below.

7.2 Summary of $\text{AdS}_3/\text{CFT}_2$ results so far

Before starting any new calculations we collect now the evidence for $\text{AdS}_3/\text{CFT}_2$ that we have obtained already in previous lectures.

- **Matching of global symmetries.** The defining property of a CFT_2 is conformal symmetry: the physical Hilbert space must fall into representations of the conformal algebra, which in two dimensions consists of two copies of the Virasoro algebra. For AdS/CFT to have any chance to be true it must be the case that the physical Hilbert space (or in the classical approximation the physical phase space) falls into representations of two copies of the Virasoro algebra. In the previous chapter we have proved that this is true for AdS_3

Einstein gravity with Brown–Henneaux boundary conditions. Starting from a QFT with Poincaré plus scale symmetries it is even possible to "derive" the AdS line element as follows. Suppose we have the daring idea to use energy E as additional coordinate, for instance to geometrize renormalization group flow of a D-dimensional QFT. The most general line-element in $D+1$ dimensions compatible with Poincaré symmetries is then given by

$$\mathrm{d}s^2 = f_1(E)\,\mathrm{d}E^2 + f_2(E)\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \qquad \mu,\nu = 0..(D-1) \tag{4}$$

with two unknown scalar functions $f_i(E)$. Suppose further that our QFT has scale symmetry

$$x^\mu \to \lambda x^\mu \qquad\qquad E \to E\lambda^{-1}\,. \tag{5}$$

Then the most general line-element (4) compatible with the scale symmetry (5) is given by

$$\mathrm{d}s^2 = \ell^2\left(\frac{\mathrm{d}E^2}{E^2} + E^2\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\right) \tag{6}$$

where we introduced some arbitrary but fixed length scale ℓ to have the correct units. The metric (6) is Poincaré patch AdS in $D+1$ dimensions, with AdS radius ℓ and the asymptotic boundary at $E\to\infty$.

- **Central charge.** Any CFT_2 is characterized, among other things, by the values of the two central charges. In the absence of gravitational or Lorentz anomalies the left and right central charges must be equal in magnitude and positive. In the previous chapter we have proved his is true for AdS_3 Einstein gravity with Brown–Henneaux boundary conditions as long as Newton's constant is positive.

- **Super-gravity limit.** From the way the string theory construction works it is clear that AdS/CFT is a duality of strong/weak type, meaning in particular that strongly coupled CFTs are mapped to weakly coupled gravity theories. Thus, we should expect that the super-gravity limit, which is very simple, should produce CFTs that are very complicated. More concretely, in the classical limit of vanishing Newton constant, $G\to 0$, the CFT central charge is expected to diverge, $c\to\infty$. In the previous chapter we have proved his is true for AdS_3 Einstein gravity with Brown–Henneaux boundary conditions.

- **Quantization of Newton's constant.** One consequence of the Zamolodchikov c-theorem (valid for CFT_2) is that in a continuous family of unitary CFTs all of them must have the same central charge. This implies that, assuming unitarity, Newton's constant cannot be a continuous parameter but must be quantized (in units of the AdS radius). We do not know an argument in the metric formulation, but in the Chern–Simons formulation discussed in chapter 4 gauge invariance of the action implies quantization of the Chern–Simons level $k = c/6 = \ell/(4G)$ and thus quantization of Newton's constant.

- **Thermodynamics.** One of the motivations for the holographic principle came from thermodynamics, since the black hole entropy scales like the area, which corresponds to the volume in one lower dimension so that the corresponding entropies can match between both sides of the holographic correspondence. Thus, it is natural to ask whether this is actually true — does the Bekenstein–Hawking entropy coincide with the entropy of the corresponding CFT? The answer depends on the type of CFT we are considering — not every CFT is expected to have a gravity dual, so we should be more specific about the type of CFT we are considering. It turns out that the type of CFT we are interested in has a sparse spectrum, meaning roughly that there are not too many operators of low conformal dimension. Such CFTs have a universal expression for their entropy in the limit of interest, known as Cardy formula. Thus, if AdS_3/CFT_2 is true the Bekenstein–Hawking entropy must be equivalent to the Cardy entropy. While we have seen this result before, we will reconsider it in more detail in the following section.

Of course, the checks above do not allow us to conclude precisely, which type of CFT_2 (if any) is dual to Einstein gravity for a certain value of Newton's constant.

There is a number of further questions that we are going to address in the remainder of this chapter:

- **Semi-classical corrections to Cardy formula.** The Cardy formula basically comes from evaluating an integral in the saddle point approximation. It is well-known and universal to obtain the leading (logarithmic) corrections to the Cardy formula, and similarly it is straightforward to calculate the leading (logarithmic) corrections to the Bekenstein–Hawking entropy. Do these results match?
- **Linearized spectrum around global vacuum.** Can we identify the vacua on both sides and derive the linearized spectra of fluctuations? And if so, do the results coincide with each other? More specifically, can we reproduce the Virasoro vacuum character through some gravity 1-loop calculation?
- **Stress tensor correlation functions.** Does the GKPW prescription actually work for the stress tensor? What happens if we calculate all stress tensor correlators in a CFT_2 (which we can, thanks to Belavin–Polyakov–Zamolodchikov) and compare them with corresponding functional derivatives of the gravity action — do all of them coincide with each other?
- **Entanglement entropy.** There is a universal expression for entanglement entropy in CFT_2 that depends only on the entangling region and the value of the central charge. Is it possible to derive this result holographically through some gravity calculation, and if so, does the answer match with the CFT_2 expression? Also, does the holographic entanglement entropy obey the various consistency relations that entanglement entropy needs to obey, like strong subadditivity?

If AdS/CFT is true the answer to all these questions must be yes.

7.3 Holographic entanglement entropy

Entanglement is a key feature of quantum mechanics and an important resource for quantum technologies such as quantum computation. As with any other physical resource we want to quantify entanglement, and there are numerous entanglement measures available. The one on which we focus is called entanglement entropy. We start with a brief definition and recap of entanglement entropy in bipartite systems and then calculate entanglement entropy for CFT_2. Next we present the Ryu–Takayanagi proposal of how to holographically calculate entanglement entropy and finally we check this proposal by holographically calculating entanglement entropy.

7.3.1 Entanglement entropy

Consider a bipartite quantum system with a direct product Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and consider a general state described by some density matrix ρ (normalized such that $\mathrm{tr}\rho = 1$). Then define the reduced density matrix as the partial trace

$$\rho_A = \mathrm{tr}_B \rho \tag{7}$$

where all degrees of freedom associated with $\mathcal{H}_B$ are traced out. Entanglement entropy (EE) is then defined as the van Neumann entropy of the reduced density matrix.

$$\boxed{S_A := -\mathrm{tr}\big(\rho_A \ln \rho_A\big)} \tag{8}$$

If there is a matrix representation of the reduced density matrix ρ_A with eigenvalues λ_i then EE can be evaluated as a corresponding sum over all eigenvalues

$$S_A = -\sum_i \lambda_i \ln \lambda_i \,. \tag{9}$$

Note that eigenvalues that are either 1 or 0 do not contribute to EE, since $1 \ln 1 = 0$ and $\lim_{\varepsilon\to 0} \varepsilon \ln \varepsilon = 0$. EE inherits the following properties inherent to the van Neumann entropy:

- unitarity: $S_A(U^\dagger \rho U) = S_A(\rho)$ for any unitary operator U
- semi-positivity: $S_A(\rho) \geq 0$, with saturation only for pure states ρ
- boundedness: $S_A \leq \ln D_A$, where D_A is the dimension of the Hilbert space $\mathcal{H}_A$, with saturation only for maximally entangled states ρ
- concavity: $S_A(\sum_i \mu_i \rho_i) \geq \sum_i \rho_i S_A(\rho_i)$ for all sets of non-negative μ_i normalized to $\sum_i \mu_i = 0$

Additionally, we have the property that $S_A = S_B$ whenever ρ is a pure state.

The simplest system of interest to illustrate entanglement entropy is a 2-qubit system, where the full Hilbert space is spanned by the four states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$. We call the first qubit 'A' and the second one 'B'. Let us consider first the direct product state

$$|\psi_1\rangle = |00\rangle = |0\rangle_A \otimes |0\rangle_B \tag{10}$$

whose density matrix $\rho = |\psi_1\rangle\langle\psi_1|$ is a 4x4 matrix with a single non-zero entry of unity at the diagonal. The reduced density matrix in this case is

$$\rho_A = \mathrm{tr}_B \rho = |0\rangle_A\, {}_A\langle 0| \tag{11}$$

which is a 2x2 matrix, again with a single nonzero entry of unity at the diagonal. EE

$$S_A = -\mathrm{tr}\big(\rho_A \ln \rho_A\big) = -\sum_i \lambda_i \ln \lambda_i = -1 \times \ln 1 - 0 \times \ln 0 = 0 \tag{12}$$

vanishes, which makes sense since we have a direct product state. (All meaningful measures of entanglement must give zero for direct product states, since these states are not entangled.) Let us consider now the pure state

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) = \frac{1}{\sqrt{2}}\big(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\big) \tag{13}$$

which is not a direct product state. The density matrix $\rho = |\psi_2\rangle\langle\psi_2|$ is again a 4x4 matrix. Reducing it by taking the trace over the 'B'-part yields

$$\rho_A = \mathrm{tr}_B \rho = \frac{1}{2}\big(|0\rangle_A\, {}_A\langle 0| + |1\rangle_A\, {}_A\langle 1|\big)\,. \tag{14}$$

Whenever the reduces density matrix is proportional to the unity matrix we call the system maximally entangled. This means that EE

$$S_A = -\mathrm{tr}\big(\rho_A \ln \rho_A\big) = -\sum_i \lambda_i \ln \lambda_i = -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \ln 2 \tag{15}$$

is maximal — in the present example it corresponds to one (e-)bit.

So in simple words, **EE counts the number of entangled bits between the subsystems A and B.** From this observation we expect that EE defined for certain entangling regions obeys an area law, $S_A \sim$ area, scaling with the area of the boundary of the entangling region.

In QFTs EE is much harder to calculate. Apart from purely technical issues (how to calculate the log of a density matrix in QFT?) there could be gauge redundancies (such as in electrodynamics) and generically we expect UV divergences. This is so, because in a QFT we have modes at arbitrarily small scales, so we need to impose a UV-cutoff and EE is expected to diverge with an appropriate power of this cutoff scale.

For an introduction into EE see the lecture notes by Daniel Harlow, 1409.1231, which are mostly geared towards addressing the information loss problem and firewalls, and the review by Calabrese and Cardy, 0905.4013, which focuses on CFT_2 results, just like the next subsection.

7.3.2 CFT_2 calculation of entanglement entropy

We are interested in EE for the situation where the subsystem A corresponds to some finite spatial region (called 'entangling region') and B to the rest of space. In this subsection we calculate EE for this setup in CFT_2.

The main tool to calculate EE by paper and pencil is the replica trick, which replaces the log of the density matrix by an integer power n (the number of 'replicas' of the QFT) and in the end analytically continues to recover the log. Thus, instead of directly calculating EE (8) one considers instead the Rényi entropies

$$S_A^{(n)} = \frac{1}{1-n}\left(\text{tr}\rho_A^n - 1\right) \tag{16}$$

in terms of which EE is given by the limit

$$S_A = \lim_{n\to 1^+} S_A^{(n)}\,. \tag{17}$$

In CFT_2, calculating the n^{th} Rényi entropy is possible essentially by evaluating a partition function on a complicated Riemann-surface or equivalently by introducing suitable twist fields, as shown below. The CFT_2 result reviewed below was first calculated by Holzhezy, Wilczek and Larsen, see hep-th/9403108.

We start now by considering a lattice version of a CFT_2 defined on a spatial lattice with spacing a (this is the UV cutoff), while keeping time τ continuous. We work in Euclidean signature but assume that we are at zero temperature for simplicity. Moreover, we assume that eventually we can take the continuum limit, and whenever possible we use this continuum limit already in the discussion. We define the subsystem A as some interval $(x_L,\, x_R)$ of length $L \gg a$. We label the complexified coordinates by $w = x + i\tau$ and $\bar{w} = x - i\tau$.

The ground state wave function ψ is given by the Euclidean path integral ranging from $\tau = -\infty$ to $\tau = 0$

$$\psi[\phi_0(x)] = \int \mathcal{D}\phi\, \exp\big(-\Gamma[\phi]\big) \tag{18}$$

evaluated with the boundary condition $\phi(\tau = 0^-,\, x) = \phi_0(x)$, where Γ is the Euclidean action and ϕ abstractly denotes the CFT_2 fields. The density matrix is then given by two copies of the ground state wave function, $\rho[\phi_0, \hat{\phi}_0] = \psi[\phi_0]\bar{\psi}[\hat{\phi}_0]$. Complex conjugation can be taken into account by integrating from $\tau = 0^+$ to $\tau = +\infty$. The reduced density matrix ρ_A is obtained by tracing out the part associated with region B, the complement of region A. We identify in the whole region B the quantities ϕ_0 and $\hat{\phi}_0$, but not in the region A, since we trace out only the B-part. This means that in the path integral we localize on the remaining region A

$$\rho_A[\phi_+,\,\phi_-] = \frac{1}{Z_1}\int_{\tau=-\infty}^{\tau=\infty}\mathcal{D}\phi\,\exp\big(-\Gamma[\phi]\big)\prod_{x\in A}\big(\delta(\phi(+0,\,x)-\phi_+)\,\delta(\phi(-0,\,x)-\phi_-)\big) \tag{19}$$

where Z_1 is the vacuum partition function on the complex plane ensuring the normalization $\mathrm{tr}\rho_A = 1$. This means that the path integral associated with ρ_A is evaluated on a 1-sheeted Riemann surface with a branch cut along the region A. In order to obtain $\mathrm{tr}\rho_A^n$ we take n copies of the result (19) and then take the trace,

$$\mathrm{tr}\rho_A^n = \rho_A[\phi_{1+},\,\phi_{1-}]\rho_A[\phi_{2+},\,\phi_{2-}]\dots\rho_A[\phi_{n+},\,\phi_{n-}] \qquad \phi_{i+} = \phi_{(i+1)-}, \phi_{n+} = \phi_{1-} \tag{20}$$

which means we have n copies of a 1-sheeted Riemann surface, and taking the trace glues them together to a single n-sheeted Riemann surface.

So to calculate the n^{th} Rényi entropy we have to evaluate the (normalized) partition function $Z_n(A)$ associated with an n-sheeted Riemann surface with a cut along the interval that defines the entangling region A, where the n sheets are glued together cyclically along these cuts (see the figures 1-3 in the Calabrese–Cardy paper):

$$\mathrm{tr}\rho_A^n = \frac{Z_n(A)}{Z^n} \tag{21}$$

The denominator is a normalization factor involving the n^{th} power of the usual partition function and ensures the correct normalization. See appendix A for how this calculation is done using twist fields $\Phi_n(x_L)$ and $\Phi_{-n}(x_R)$, which are CFT_2 primary operators with the same conformal weights $\Delta_n = \bar{\Delta}_n$ displayed in (53) at the end of the appendix.

The expression in (21) is given by the n^{th} power of the 2-point function of these twist fields.

$$\mathrm{tr}\rho_A^n = \Big(\langle\Phi_n(x_L)\Phi_{-n}(x_R)\rangle\Big)^n \propto |x_L - x_R|^{-4n\Delta_n} \tag{22}$$

This means that up to an overall normalization constant N_n we obtain

$$\mathrm{tr}\rho_A^n = N_n\left(\frac{L}{a}\right)^{-c(n-1/n)/6} \tag{23}$$

where we inserted the cutoff scale a in order to make the normalization constant N_n dimensionless (note that $N_1 = 1$). The final result (23) can easily be continued analytically to real values of $n \geq 1$.

Plugging the analytically continued result (23) for the n^{th} Rényi entropy into the limit (17) for EE establishes

$$\boxed{\text{EE for planar CFT}_2\text{ at zero temperature :}\qquad S_A = \frac{c}{3}\,\ln\frac{L}{a} + \text{const.}} \tag{24}$$

In the result (24) c is the central charge of the CFT (assuming here that left and right central charges are equal to each other), L is the size of the entangling region defining the subsystem A and a is the UV cutoff (e.g. some lattice spacing). The L-independent additive constant is non-universal and depends on the first n-derivative of the normalization constant N_n at $n = 1$ as well as on the specific choice of the cutoff scale; it does not play any role in our discussion.

The result for EE (24) is universal, in the sense that it does not depend on any details of the CFT_2 other than the central charge. Using conformal maps it is straightforward to generalize the result (24), e.g. to finite temperature or CFT's defined on the cylinder instead of on the plane.

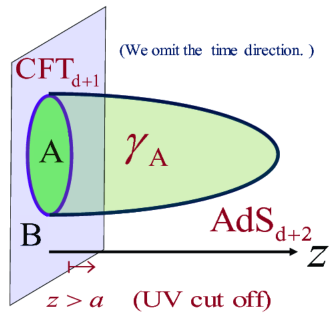

Figure 1: RT prescription for holographic EE

7.3.3 Ryu–Takayanagi proposal

The proposal of Ryu and Takayanagi (RT) hep-th/0603001 is to calculate EE holographically by the following recipe. For any entangling region in the CFT take a minimal surface γ_A attached to the boundary defining the entangling region A. Its area gives EE. The RT-formula

$$S_{\text{RT}} = \frac{\text{area}(\gamma_A)}{4G} \tag{25}$$

resembles the Bekenstein–Hawking formula, but note that the latter is a thermal entropy (not EE) and involves the area of the event horizon of a black hole (not a minimal area hanging from some asymptotically AdS boundary).

Originally, RT was checked only for time-independent situations. In time-dependent situations Hubeny, Rangamani and Takayanagi (HRT) generalized the proposal 0705.0016 with the result that minimal surfaces are replaced by extremal surfaces. The (H)RT proposal applies to any spacetime dimension. In the $\text{AdS}_3/\text{CFT}_2$ context it simplifies to calculating the length of geodesics, which is a rather straightforward calculation. See figure 1.

One of the many neat aspects of the RT proposal (apart from its simplicity) is that it allows to prove straightforwardly some inequalities that otherwise are harder to prove. For instance, the strong subadditivity inequality

$$S_{A\cup B} + S_{B\cup C} \geq S_{A\cup B\cup C} + S_B \tag{26}$$

is immediately evident from figure 2, just from knowing that EE corresponds to the area of minimal surfaces (see 0704.3719).

7.3.4 AdS_3 calculation of holographic entanglement entropy

The dual geometry to the CFT_2 vacuum on the plane is given by Poincaré patch AdS.

$$\text{d}s^2 = \frac{\ell^2}{z^2}\left(-\text{d}t^2 + \text{d}x^2 + \text{d}z^2\right) \tag{27}$$

We apply now the RT prescription (25) to this case for an entangling region of size L, i.e., the endpoints of the geodesic are ($z_L = 0$, $x_L = -L/2$) and ($z_R = 0$, $x_R = L/2$). Since $\int^0 \text{d}z/z = \infty$ the length of the geodesic diverges, which recovers the expected UV divergence of the CFT result (24) in the limit $a \to 0$. To introduce the analogue of the UV cutoff on the gravity side we anchor the geodesics not at $z = 0$ but instead

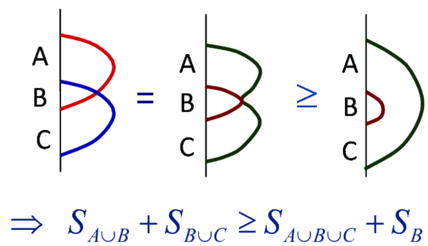

Figure 2: Holographic proof by inspection of strong subadditivity. The red and blue curves on the left are geodesics. By reinterpreting the curves (see the change of colors) they no longer are geodesics for the entangling regions depicted on the right of the figure.

at $z=\varepsilon$, with some small but finite cutoff ε. EE is thus given by

$$S_A = \frac{1}{4G}\int \mathrm{d}s = \frac{\ell}{2G}\int\limits_{\varepsilon}^{z^{\max}} \frac{\mathrm{d}z}{z}\,\sqrt{x'^{\,2}+1} = \frac{\ell}{2G}\int\limits_{L/2-\mathcal{O}(\varepsilon)}^{0} \frac{\mathrm{d}x}{z}\,\sqrt{1+\dot z^2} = \frac{\ell}{2G}\int\limits_{L/2-\mathcal{O}(\varepsilon)}^{0} \mathrm{d}x\,\mathcal{L}(z,\,\dot z) \tag{28}$$

where $z^{\max}$ is the maximal value of z, i.e., the point where the geodesic turns around back towards the asymptotic boundary. Prime denotes z-derivatives and dot x-derivatives. We choose the parametrization in terms of x.

There is a Noether charge due to invariance under x-translations

$$Q = \mathcal{L} - \dot z\,\frac{\partial\mathcal{L}}{\partial\dot z} = \frac{\ell}{z}\,\frac{1}{\sqrt{1+\dot z^2}} \tag{29}$$

which is related to the maximal z-value (where $\dot z_{\max}=0$) through

$$Q = \frac{\ell}{z_{\max}}\,. \tag{30}$$

We can also relate it to the interval length.

$$L/2-\mathcal{O}(\varepsilon) = \int\limits_0^{L/2-\mathcal{O}(\varepsilon)} \mathrm{d}x = \int\limits_{z_{\max}}^{\varepsilon} \frac{\mathrm{d}z}{\dot z} = z_{\max}\sqrt{1-\varepsilon^2} = z_{\max} - \mathcal{O}(\varepsilon^2) \tag{31}$$

The length integral (28) then simplifies to

$$S_A = \frac{\ell}{2G}\int\limits_{\varepsilon/z_{\max}}^{1} \frac{\mathrm{d}y}{y}\,\frac{1}{\sqrt{y^2-1}} = \ln\frac{z_{\max}}{\varepsilon} + \mathcal{O}(\varepsilon^2\ln\varepsilon)\,. \tag{32}$$

Labelling the UV cutoff as $\varepsilon \propto a$ and using the relation (31) the final result for holographic entanglement entropy

$$S_A = \frac{\ell}{2G}\,\ln\frac{L}{a} + \text{const.} = \frac{c}{3}\,\ln\frac{L}{a} + \text{const.} \tag{33}$$

reproduces precisely the CFT_2 result (24) for any length L and central charge c.

7.3.5 Generalization to all states dual to Bañados geometries

Above we have derived holographically EE for CFT_2 states dual to Poincaré patch AdS_3. Using the fact that any Bañados geometry (vacuum solutions of AdS_3 Einstein gravity) can be locally mapped to Poincaré patch AdS_3 the result can be generalized to EE for CFT_2 states dual to arbitrary Bañados geometries, including thermal AdS_3, global AdS_3, BTZ black holes and their Virasoro descendants. The Bañados geometries are labelled by a holomorphic and an antiholomorphic function, $\mathcal{L}^\pm(x^\pm)$, see e.g. Eq. (43) in chapter 4. The final result of these calculations for holographic EE yields (see 1605.00341)

$$S_A = \frac{c}{6}\,\ln\left(\frac{\ell^+(x_1^+,\,x_2^+)\ell^-(x_1^-,\,x_2^-)}{a^2}\right) + \text{const.} \tag{34}$$

where a is again a UV cutoff, c is the central charge, $x_1^\pm$ and $x_2^\pm$ are the two endpoints defining the entangling region and the functions $\ell^\pm$ are bilinears of other functions $\psi_{1,2}^\pm$.

$$\ell^\pm(x_1^\pm, x_2^\pm) = \psi_1^\pm(x_1^\pm)\psi_2^\pm(x_2^\pm) - \psi_2^\pm(x_1^\pm)\psi_1^\pm(x_2^\pm) \tag{35}$$

The functions $\psi_{1,2}^\pm$ are two independent solutions to Hill's equation

$$\psi^{\pm\prime\prime} - \mathcal{L}^\pm\,\psi^\pm = 0 \tag{36}$$

with unit Wronskian, $\psi_2^\pm\psi_1^{\pm\prime} - \psi_1^\pm\psi_2^{\pm\prime} = \pm 1$.

As a sanity check, let us recover first from above the Poincaré patch result (33). In that case $\mathcal{L}^\pm = 0$ and the normalized solutions to Hill's equation read $\psi_1^+ = x^+$, $\psi_2^+ = 1 = \psi_1^-$ and $\psi_2^- = x^-$. For a constant time slice we have $|x_1^+ - x_2^+| = |x_1^- - x_2^-| = L$ and thus the general result (34) yields

$$\text{Poincaré}: \qquad S_A = \frac{c}{6}\,\ln\left(\frac{|x_1^+ - x_2^+||x_1^- - x_2^-|}{a^2}\right) + \text{const.} = \frac{c}{6}\,\ln\left(\frac{L^2}{a^2}\right) + \text{const.} \tag{37}$$

which coincides precisely with (33).

For BTZ black holes we have constant $\mathcal{L}^\pm \geq 0$ and the appropriate solutions to Hill's equation read

$$\psi_1^\pm = \frac{1}{\sqrt{2\sqrt{\mathcal{L}^\pm}}}\,e^{\sqrt{\mathcal{L}^\pm}x^\pm} \qquad\qquad \psi_2^\pm = \frac{1}{\sqrt{2\sqrt{\mathcal{L}^\pm}}}\,e^{-\sqrt{\mathcal{L}^\pm}x^\pm}\,. \tag{38}$$

Assuming again an equal time entangling region of length L inserting (38) into (34)-(35) yields (we drop from now on the trivial additive constant to EE)

$$\text{BTZ}: \qquad S_A = \frac{c}{6}\,\ln\left(\frac{\sinh(\sqrt{\mathcal{L}^+}L)\sinh(\sqrt{\mathcal{L}^-}L)}{\sqrt{\mathcal{L}^+\mathcal{L}^-}a^2}\right). \tag{39}$$

The simpler case of non-rotating BTZ black holes, $\mathcal{L}^+ = \mathcal{L}^- = \pi^2T^2$ (with T being the Hawking temperature, see chapter 4 or Black Holes II), yields

$$\text{non-rotating BTZ}: \qquad S_A = \frac{c}{3}\,\ln\left(\frac{\sinh\left(\pi T\,L\right)}{\pi T\,a}\right) \tag{40}$$

which coincides precisely with the EE for thermal states in a CFT_2 at temperature T, see 0905.4013. The small temperature limit $T \to 0$ reproduces the Poincaré patch result (33), as expected, while the high temperature limit yields a volume law

$$\lim_{T\to\infty} S_A = \frac{c}{3}\,\pi T\,L + \dots \tag{41}$$

7.3.6 Further developments

Starting with the seminal RT-proposal the last 13 years have brought many new developments merging the concepts of quantum gravity and quantum information via holography, often using (holographic) EE as key player. This includes proofs of c- and F-theorems, conjectures and proofs of quantum energy conditions, connections to MERA, the firewall puzzle, bulk reconstruction from boundary data, constructions of the interior of black holes, emergence of apparent information loss in CFT_2, a bound on chaos, holographic complexity and generalizations of holographic EE beyond AdS/CFT, e.g. to flat space holography. Here is a (rather incomplete) list of reviews and lecture notes to guide you through these exciting recent developments:

- Tatsuma Nishioka, Shinsei Ryu and Tadashi Takayanagi, "Holographic Entanglement Entropy", https://arxiv.org/abs/0905.0932
- Daniel Harlow, "Jerusalem Lectures on Black Holes and Quantum Information", https://arxiv.org/abs/1409.1231
- Thomas Hartman, "Lectures on Quantum Gravity and Black Holes", http://www.hartmanhep.net/topics2015/gravity-lectures.pdf
- Horacio Casini, Marina Huerta, Robert Myers and Alexandre Yale, "Mutual information and the F-theorem", https://arxiv.org/pdf/1506.06195.pdf
- Arjun Bagchi, Rudranil Basu, Ashish Kakkar and Aditya Mehra, "Flat holography: Aspects of the dual field theory", https://arxiv.org/abs/1609.06203
- Edward Witten, "A Mini-Introduction To Information Theory", https://arxiv.org/abs/1805.11965
- Leonard Susskind, "Three Lectures on Complexity and Black Holes", https://arxiv.org/abs/1810.11563

7.4 Cardy formula

... to be done ...

7.5 Linearized spectrum around global vacuum

... to be done

7.6 Stress tensor correlation functions

... to be done ...

7.7 Open questions and outlook

Of course, there are still numerous questions that we have not addressed, even after answering all the questions above in the affirmative. There is still a lot of work to be done before we can claim to fully understand AdS/CFT! Hopefully, it became evident why most of us tend to believe that AdS/CFT is correct — if AdS/CFT was incorrect then all the checks performed in this chapter and the numerous additional checks not discussed here would be due to coincidences, and by the sheer amount of checks this seems unlikely.[1]

[1] While we have no proof that AdS/CFT is correct, we also have no proof that the Standard Model exists and yet find good use for it in physics. In fact, the only proof we have in this regard is that the Standard Model does *not* exist, so proofs and no-go's are sometimes a bit

A Rényi entropies in CFT_2 from twist fields

Here is a sketchy summary of how to calculate Rényi entropies in CFT_2.

By virtue of (20) the calculation of $Z_n(A)$ is done efficiently via so-called twist fields

$$Z_n(A) \propto \langle \Phi_n(x_{L1})\Phi_{-n}(x_{R1})\dots\Phi_n(x_{Ln})\Phi_{-n}(x_{Rn})\rangle_{\mathbb{C}} \tag{42}$$

where the right hand side involves a correlation function on the complex plane. In path integral language the equation above reads

$$Z_n(A) = \int_{\text{b.c.}} \mathcal{D}\phi_1 \dots \mathcal{D}\phi_n \exp\Big[-\int_{\mathbb{C}} \mathrm{d}\tau\,\mathrm{d}x\big(\mathcal{L}(\phi_1)+\dots\mathcal{L}(\phi_n)\big)\Big] \tag{43}$$

with the boundary conditions

$$\phi_i(0^+,\,x) = \phi_{i+1}(0^-,\,x) \qquad \phi_n(0^+,\,x) = \phi_1(0^-,\,x) \qquad\qquad \forall x \in A\,. \tag{44}$$

By $\Phi_n(x_{Li})$ we mean the twist field that is associated with the cyclic permutation symmetry $j \to j+1 \bmod n$ evaluated on the i^{th} sheet of the Riemann surface at the left location of the branch cut, while $\Phi_n(x_{Ri})$ is the same for the right location of the branch cut. Changing n to $-n$ corresponds to the inverse cyclic permutation symmetry $j+1 \to j \bmod n$.

The twist fields allow to map the calculation of expectation values of some operator $\mathcal{O}$ for the theory on the n-sheeted Riemann surface into a calculation of correlations functions with twist field insertions for the theory on the complex plane.

$$\langle\mathcal{O}(\tau,\,x)\rangle_{n\text{-Riemann}} = \frac{\langle\Phi_n(x_L)\Phi_{-n}(x_R)\mathcal{O}(\tau,\,x)\rangle_{\mathbb{C}}}{\langle\Phi_n(x_L)\Phi_{-n}(x_R)\rangle_{\mathbb{C}}} \tag{45}$$

In the following we exploit this result by choosing for $\mathcal{O}$ the stress tensor, since its transformation behavior under conformal maps (in our case from the n-sheeted Riemann surface with branch cut to the complex plane) is well-known.

There is a standard way to map the n-sheeted Riemann surface with branch cuts to the complex plane. First, we map the interval to $(0,\,\infty)$ via the conformal map $w \to \zeta = (w-x_L)/(w-x_R)$. Then we uniformize using the root-function, $\zeta \to z = \zeta^{1/n} = ((w-x_L)/(w-x_R))^{1/n}$, so that z is defined on the $\mathbb{C}$-plane without cuts.

Let us now check how the stress tensor transforms under this conformal map. The general transformation law of the stress tensor under conformal maps

$$T(w) = \Big(\frac{\mathrm{d}z}{\mathrm{d}w}\Big)^2 T(z) + \frac{c}{12}\,\{z,\,w\}_{\text{Sch}} \tag{46}$$

involves the Schwarzian derivative

$$\{z,\,w\}_{\text{Sch}} := \frac{z'''}{z'} - \frac{3}{2}\,\frac{z''^2}{z'^2} \tag{47}$$

whose infinitesimal version we have encountered already on the gravity side,

$$\delta_\varepsilon T(w) = T'(w)\,\varepsilon(w) + 2T(w)\,\varepsilon(w)' + \frac{c}{12}\,\varepsilon(w)''' \tag{48}$$

where we inserted $z = w + \varepsilon(w)$ into the finite transformation law (46) and neglected terms of $\mathcal{O}(\varepsilon^2)$. The defining property of the Schwarzian derivative is that

overrated when it comes to physics applications (not because of the concept of proof per se, which is obviously powerful and needed, but because in most proofs and no-go's there are some hidden or explicit assumptions that physical systems may circumvent). In fact, one could define a theoretical physicist's job description as "breaker of no-go's".

it is annihilated for any fractional linear transformation, $z = (aw+b)/(cw+d)$, corresponding to global conformal transformations.

Applying the finite transformation law (46) to our case and taking expectation values yields

$$\langle T(w)\rangle_{n\text{-Riemann}} = \frac{c}{12}\,\{z,\,w\}_{\text{Sch}} = \frac{c(n^2-1)}{24n^2}\,\frac{(x_L-x_R)^2}{(w-x_L)^2(w-x_R)^2} \tag{49}$$

where we used $\langle T(z)\rangle_{\mathbb{C}} = 0$.

With hindsight (or the appropriate CFT_2 insight), the right hand side of (49) looks like a normalized 3-point function of the stress tensor and some primary fields Φ_n, Φ_{-n} with a certain scaling dimension, evaluated on the complex plane,

$$\langle T(w)\,\Phi_n(u)\,\Phi_{-n}(v)\rangle_{\mathbb{C}} = \frac{\Delta_n}{(w-u)^2(w-v)^2(u-v)^{2\Delta_n-2}(\bar{u}-\bar{v})^{2\bar{\Delta}_n}} \tag{50}$$

with the normalization

$$\langle\Phi_n(u)\,\Phi_{-n}(v)\rangle_{\mathbb{C}} = |u-v|^{-2\Delta_n-2\bar{\Delta}_n}\,. \tag{51}$$

The primary fields Φ_n, Φ_{-n} are known as "twist fields". Identifying $u = x_L$ and $v = x_R$ and dividing the 3-point function (50) by the normalization (51) yields

$$\frac{c(n^2-1)}{24n^2}\,\frac{(x_L-x_R)^2}{(w-x_L)^2(w-x_R)^2} = \frac{\langle T(w)\,\Phi_n(x_L)\,\Phi_{-n}(x_R)\rangle_{\mathbb{C}}}{\langle\Phi_n(x_L)\Phi_{-n}(x_R)\rangle_{\mathbb{C}}} \tag{52}$$

provided we fix the scaling dimensions of the twist fields as

$$\Delta_n = \bar{\Delta}_n = \frac{c(n^2-1)}{24n^2}\,. \tag{53}$$