5 Gravity in two spacetime dimensions

Two is the lowest dimensions that allows for black hole horizons, since you need at least one coordinate with positive and one with negative signature. This provides the main motivation to consider two-dimensional toy-models of classical and quantum gravity.

An additional motivation comes from extremal black holes. Typically, they have an AdS_2 factor, so it seems natural to try to apply holography to this factor, and for this we need some two-dimensional model of gravity.

Finally, even classically it can be a technical expedient to consider dimensional reductions of higher-dimensional gravity theories to two dimensions — for instance, spherical reduction of D-dimensional Einstein gravity allows for a quick derivation of the Schwarzschild–Tangherlini or Reissner–Nordsström–(A)dS solutions in a few lines. So let us delve right into two-dimensional gravity.

At a technical level an important property/simplification of two-dimensional gravity is that any metric is locally conformally flat (this is not true in three or higher dimensions), so that locally one can employ conformal gauge

$$\mathrm{d}s^2 = e^{2\Omega(x^\mu)}\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \qquad\qquad \mu,\,\nu\in\{0,\,1\} \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. It is often convenient to use light-cone coordinates for the Minkowski metric, $\eta_{\pm\mp}=1$, $\eta_{\pm\pm}=0$.

5.1 No Einstein gravity

Einstein gravity is the simplest theory of gravity, but it does not exist in two dimensions. There are several ways to see this. The simplest one is to observe that the Einstein equations $R_{\mu\nu}=\frac{1}{2}\,g_{\mu\nu}\,R$ are purely kinematical in two dimensions and hold for any metric, regardless of field equations. Equivalently, the Einstein–Hilbert action (in presence of smooth boundaries)

$$\frac{\kappa}{4\pi}\int_{\mathcal{M}}\mathrm{d}^2x\sqrt{|g|}\,R+\frac{\kappa}{2\pi}\int_{\partial\mathcal{M}}\mathrm{d}x\sqrt{|\gamma|}\,K=\kappa\,\chi(\mathcal{M}) \tag{2}$$

yields the Euler characteristic $\chi(\mathcal{M})$ and varies to zero. Finally, yet-another way to see that Einstein gravity in two dimensions is not kosher is by extrapolating the count of gravitational degrees of freedom, $D(D-3)/2$, to two dimensions, which yields -1. This last observation, while meaningless at face value, suggests at least a possible resolution: if we want to have a theory similar to three dimensions — with zero local degrees of freedom, but non-trivial equations of motion and potentially globally non-trivial solutions — then we should add a degree of freedom. The simplest way to do this is by adding a scalar field.

5.2 Various attempts

Given that Einstein gravity does not exist in two dimensions we have to look for alternative theories of gravity, deviating as little as possible from Einstein gravity. We describe now various ways to do this, all leading to the same type of theory.

5.2.1 Gravity as gauge theory

In three dimensions the reformulation of Einstein gravity as non-abelian gauge theory of Chern–Simons type turned out to be quite useful. The key feature of Chern–Simons theories is that the equations of motion imply gauge flatness. There is a similar theory in two dimensions, namely non-abelian BF-theory, with action

$$I_{\text{BF}} = \frac{k}{2\pi} \int \langle B\, F \rangle \tag{3}$$

where B is some coadjoint scalar, $F = \mathrm{d}A + A \wedge A$ the non-abelian field strength associated with some gauge connection A, and $\langle , \rangle$ denotes again the bilinear form.

The action (3) is meaningful since it is given by an integral over a gauge-invariant 2-form. The equations of motion obtained by varying with respect to B establish gauge flatness, $F = 0$, as desired. So far so good — but BF-theories are not generically gravity theories, in the same way that Chern–Simons theories are not generically gravity theories. We need to choose a special gauge group and identify the Cartan variables within the connection A.

By analogy to three dimensions we postulate the connection

$$A = e^a\, P_a + \omega\, J \tag{4}$$

where we dualized the spin-connection ($\sigma = \pm 1$, depending on signature)

$$\omega^{ab} = \sigma \epsilon^{ab}\, \omega\, . \tag{5}$$

The translations P_a and boost J generators obey the commutation relations associated with (A)dS$_2$ or flat space, depending on the sign of Λ

$$[P_a,\, P_b] = -\Lambda\, \epsilon_{ab}\, J \tag{6}$$

$$[P_a,\, J] = \epsilon_a{}^b\, P_b \tag{7}$$

where the flat indices are raised and lowered with the Minkowski metric η_{ab}. The B-field expands similarly.

$$B = X^a\, P_a + X\, J \tag{8}$$

The bilinear form is given by

$$\langle P_a,\, P_b \rangle = \eta_{ab} \qquad\qquad \langle J,\, J \rangle = 1\, . \tag{9}$$

Inserting (4)-(9) into the BF-action yields the **Jackiw–Teitelboim (JT) model**

$$I^{(1)}_{\text{JT}}[X,\, X^a,\, \omega,\, e^a] = \frac{k}{2\pi} \int \big(X_a T^a + X\, (\mathrm{d}\omega - \Lambda\, \epsilon) \big) \tag{10}$$

where $T^a = \mathrm{d}e^a + \sigma \epsilon^a{}_b \omega \wedge e^b$ is the torsion 2-form and $\epsilon = \frac{1}{2}\, \epsilon_{ab} e^a \wedge e^b$ is the volume 2-form with $*\epsilon = **1 = \sigma$.

Variation of the JT bulk action (10) with respect to the Lagrange multipliers X_a establishes vanishing torsion on-shell, reminiscent of what happens in Einstein–Hilbert–Palatini. Variation with respect to the so-called **dilaton field** X shows that all solutions are of constant curvature, $R = 2\sigma * \mathrm{d}\omega = 2\sigma\Lambda * \epsilon = 2\Lambda$.

Eliminating the connection by solving the condition of vanishing torsion (and defining $\frac{1}{2}\, \epsilon_{ab}\, e^a \wedge e^b =: -\sigma\, \mathrm{d}^2 x \sqrt{\sigma g}$) yields the second order formulation of the JT model

$$I^{(2)}_{\text{JT}}[X,\, g_{\mu\nu}] = -\sigma\, \frac{k}{4\pi} \int \mathrm{d}^2 x \sqrt{\sigma g}\, X\, \big(R - 2\Lambda \big) \tag{11}$$

where R is now the Ricci scalar. The only difference to the Einstein–Hilbert action is the presence of an additional scalar field, the dilaton X. It can be interpreted as a spacetime-dependent Newton constant or, alternatively, as a modification of the volume-form. Such theories are known as scalar-tensor theories or **dilaton gravity**.

5.2.2 Higher curvature theory and/or torsion

A few years after the JT model appeared, **Katanaev and Volovich** considered gravity as Poincaré-gauge theory. To keep things simpler we study a simplified version of their model where torsion is zero and use Minkowski signature. The key idea is to introduce an extra degree of freedom by considering non-linearities in curvature (and possibly torsion). The simplest such model has an action quadratic in curvature

$$I_{R^2}[g_{\mu\nu}] = \frac{k}{8\pi} \int \mathrm{d}^2x\sqrt{-g}\, R^2\,. \tag{12}$$

It is straightforward to convert this action into one that is more similar to (11):

$$I^{(2)}_{R^2}[X,\, g_{\mu\nu}] = \frac{k}{4\pi} \int \mathrm{d}^2x\sqrt{-g}\,\big(XR - \tfrac{1}{2}\,X^2\big)\,. \tag{13}$$

Again, we ended up with an action that contains a scalar field X in addition to the metric $g_{\mu\mu}$.

The corresponding first order action reads

$$I^{(1)}_{R^2}[X,\, X^a,\, \omega,\, e^a] = \frac{k}{2\pi} \int \big(X_aT^a + X\;\mathrm{d}\omega - \tfrac{1}{4}\,X^2\,\epsilon\big)\,. \tag{14}$$

The generalization to the Katanaev–Volovich model allows also for bilinear terms in the Lagrange-multipliers for torsion and a cosmological constant,

$$I^{(1)}_{\text{KV}}[X,\, X^a,\, \omega,\, e^a] = \frac{k}{2\pi} \int \big(X_aT^a + X\;\mathrm{d}\omega + \big(\lambda + \alpha X^2 + \beta X_aX^a\big)\,\epsilon\big)\,. \tag{15}$$

5.2.3 Strings in two dimensions

We will be more sketchy in this section — for a slightly longer exposition see section 2.1.2 in the rveiew hep-th/0204253. (Bosonic) closed string theory can be defined in terms of a non-linear σ-model worldsheet action, involving as background fields the target-space metric $g_{\mu\nu}$, the dilaton field Φ, and possibly a B-field, which we set to zero for simplicity. Local scale invariance implies the vanishing of the β-functions associated with this σ-model, which can be re-interpreted as equations of motion descending from some target space action. For the special case when the target space is 2-dimensional (which is an example of non-critical string theory) this target space action reads

$$I^{(2)}_{\text{target}}[\Phi,\, g_{\mu\nu}] = \frac{k}{4\pi} \int \mathrm{d}^2x\sqrt{-g}\, e^{-2\Phi}\,\big(R + 4(\nabla\Phi)^2 - 4\lambda^2\big) \tag{16}$$

where $\lambda^2 \propto 1/\alpha'$, with α' being the string tension.

Redefining the dilaton field $X = e^{-2\Phi}$ we have again an action similar to the ones considered before, but now with kinetic term for the dilaton (again $\sigma = \pm 1$, depending on signature),

$$I^{(2)}[X,\, g_{\mu\nu}] = -\sigma\,\frac{k}{4\pi} \int \mathrm{d}^2x\sqrt{\sigma g}\,\big(XR - \sigma U(X)(\nabla X)^2 - 2V(X)\big) \tag{17}$$

with $U(X) = -1/X$ and $V(X) = 2\lambda^2 X$. The corresponding first order action is given by

$$I^{(1)}[X,\, X^a,\, \omega,\, e^a] = \frac{k}{2\pi} \int \big[X_aT^a + X\;\mathrm{d}\omega - \big(V(X) + \tfrac{\sigma}{2}\,X^aX_aU(X)\big)\,\epsilon\big]\,. \tag{18}$$

5.2.4 Symmetry reductions of higher-dimensional Einstein gravity

Let us start with the Einstein–Hilbert action in D dimensions and assume spherical symmetry,

$$\mathrm{d}s^2 = g^{(D)}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + \frac{1}{\lambda}\,X^{2/(D-2)}\,\mathrm{d}^2\Omega_{S^{D-2}} \tag{19}$$

where $\alpha, \beta = 0, 1$ while μ, ν range over the whole dimension, $\mathrm{d}^2\Omega_{S^{D-2}}$ is the line-element of the round S^{D-2}, and the two-dimensional metric $g_{\alpha\beta}$ as well as the surface radius X depend on x^α only.

Inserting the ansatz (19) into the Einstein–Hilbert action and integrating out the angular part yields a two-dimensional action (17) with the relevant functions given by

$$U(X) = -\frac{D-3}{D-2}\,\frac{1}{X} \qquad\qquad V(X) = -\frac{\lambda}{2}\,(D-2)(D-3)\,X^{\frac{D-4}{D-2}}\,. \tag{20}$$

This means that the s-wave sector of General Relativity in any spacetime dimension greater than two can be described by a two-dimensional dilaton gravity theory.

There are two interesting special cases. In the limit of infinite dimension, $D \to \infty$, the potentials (20) (upon suitably rescaling λ) tend to the ones of string theory in two dimensions, see 1303.1995 for a discussion. In Weinberg's limit of $2+\epsilon$ dimensions after a trick that uses some duality the so-called **Liouville action** is obtained, see 0712.3775.

$$I^{(2)}_{\text{Liouville}} = \frac{k}{4\pi}\int \mathrm{d}^2x\sqrt{-g}\,\big(XR - (\nabla X)^2 - \lambda e^{-2X}\big) \tag{21}$$

As we shall see in the next section, the action (21) is the closest analogue of Einstein gravity in two dimensions, as it is one of the few dilaton gravity theories that obeys Wheeler's dictum "matter tells geometry how to curve".

5.3 Dilaton gravity

In the previous section we discussed five different attempts to come up with a gravity model in two dimensions, and it turned out that all of them led to the same type of action, namely dilaton gravity. Thus, it makes sense to focus on generic dilaton gravity and discuss its main properties in a way that is as model-independent as possible.

The full dilaton gravity action for Minkowski signature ($\sigma = -1$) is given by

$$\Gamma^{(2)}[X,\, g_{\mu\nu}] = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{d}^2x\sqrt{-g}\,\big(XR + U(X)(\nabla X)^2 - 2V(X)\big) + \frac{k}{2\pi}\int_{\partial\mathcal{M}} \mathrm{d}x\sqrt{-\gamma}\,\big(XK - S(X)\big) \tag{22}$$

and is characterized by two arbitrary functions $U(X)$ and $V(X)$. The boundary terms comprise a straightforward generalization of the GHY boundary term and a holographic counterterm given by the so-called pre-potential

$$S(X) := \Big(-2e^{-\int^X \mathrm{d}y U(y)}\int^X \mathrm{d}y\, V(y)\, e^{\int^y \mathrm{d}z U(z)}\Big)^{1/2}. \tag{23}$$

Given some "natural" assumptions on the allowed variations of the dilaton field (in particular $\frac{1}{X}\,\delta X = 0$) the action (22) with (23) has a well-defined variational principle, see hep-th/0703230 for details. A relatively extensive list of models is contained in table 1 of hep-th/0604049 (beware of conventions!).

Similar to Einstein gravity, the equations of motion are given by second order partial differential equations:

$$R - U'(\partial X)^2 - 2U\Box X - 2V' = \frac{4\pi}{k}\,\hat{T} \tag{24a}$$

$$2U(\partial_\mu X)(\partial_\nu X) - g_{\mu\nu}U(\partial X)^2 + 2g_{\mu\nu}V - 2\nabla_\mu\partial_\nu X + 2g_{\mu\nu}\Box X = \frac{4\pi}{k}\,T_{\mu\nu} \tag{24b}$$

The right hand sides are non-zero in the presence of additional matter degrees of freedom given by some matter action $I_{\textrm{mat}}$.

$$T_{\mu\nu} := \frac{2}{\sqrt{-g}}\,\frac{\delta I_{\textrm{mat}}}{\delta g^{\mu\nu}} \qquad\qquad \hat{T} := \frac{1}{\sqrt{-g}}\,\frac{\delta I_{\textrm{mat}}}{\delta X} \tag{25}$$

Taking the trace of the second equation of motion yields $4V + 2\Box X = \frac{4\pi}{k}\,T$, which allows to express the first equation of motion as

$$R = U'(\partial X)^2 - 4UV + 2V' + \frac{4\pi}{k}\,\big(\hat{T} + UT\big)\,. \tag{26}$$

Thus, for general dilaton gravity models spacetime can have curvature even in the absence of matter, $T = \hat{T} = 0$. The only exceptions are dilaton gravity models whose potentials obey the differential equation

$$U'(\partial X)^2 - 4UV + 2V' = 0 \tag{27}$$

and hence have the "Wheeler property" that "matter tells spacetime how to curve". Since X is in general spacetime-dependent, (27) yields two conditions, $U' = 0$ and $V' = 2UV$. They are solved either by $U = 0$ and $V = \textrm{const.}$ (known as **CGHS model**, see hep-th/9111056) or by $U = -\alpha = \textrm{const.}$ and $V \propto e^{-2\alpha X}$ (Liouville gravity, see end of section 5.2.4).

There is a lot more that we can and will say about the classical theory and its solution space, but before doing so we introduce the gauge theoretic formulation of two-dimensional dilaton gravity, in terms of which many of these statements are far easier to obtain than in the metric formulation.

5.4 Poisson-σ model formulation

Like in three dimensions let us now switch gears for a while and consider a specific topological gauge theory, namely a generalization of non-abelian BF-theory known as **Poisson-$\boldsymbol{\sigma}$ model (PSM)**, see hep-th/9312059 and hep-th/9405110. Its bulk action is given by

$$I_{\textrm{PSM}}[X^I,\,A_I] = \frac{k}{2\pi}\int\big(X^I\,\mathrm{d}A_I + \tfrac12\,P^{IJ}(X^K)\,A_I\wedge A_J\big) \tag{28}$$

where X^I are some co-adjoint scalars, A_I is a connection 1-form and $P^{IJ} = -P^{JI}$ an anti-symmetric tensor called "Poisson tensor", subject to the non-linear Jacobi identity

$$P^{IL}\partial_L P^{JK} + P^{JL}\partial_L P^{KI} + P^{KL}\partial_L P^{IJ} = 0\,. \tag{29}$$

Defining the Schouten–Nijenhuis bracket $\{X^I,\,X^J\} := P^{IJ}$ the identity (29) is literally the Jacobi identity for this bracket. One can interpret the scalars X^I as target space coordinates of a Poisson-manifold.[1]

[1]Poisson manifolds are generalizations of symplectic manifolds where the Poisson tensor need not be invertible.

The PSM action enjoys gauge invariance under non-linear gauge symmetries,

$$\delta_\lambda X^I = P^{IJ}\,\lambda_J \tag{30}$$

$$\delta A_I = -\,\mathrm{d}\lambda_I - \left(\partial_I P^{JK}\right) A_J\,\lambda_K \tag{31}$$

which reduce to abelian gauge symmetries if the Poisson tensor is constant and to Yang–Mills type of gauge symmetries if the Poisson tensor is linear in the target space coordinates.

The equations of motion

$$\mathrm{d}X^I + P^{IJ}\,A_J = 0 \tag{32}$$

$$\mathrm{d}A_I + \tfrac{1}{2}\left(\partial_I P^{JK}\right) A_J \wedge A_K = 0 \tag{33}$$

allow for two classes of solutions, which with hindsight we shall refer to as "constant dilaton vacua" and "linear dilaton vacua".

The constant dilaton vacua are solutions where the target space coordinates are constant and the Poisson tensor vanishes,

$$X^I = \bar{X}^I = \text{const.} \qquad\qquad P^{IJ}(\bar{X}^K) = 0 \tag{34}$$

In this case the first equation of motion (33) holds trivially, while the second one (33) establishes gauge flatness of the connection. On constant dilaton vacua the rank of the Poisson tensor is zero.

The linear dilaton vacua lead to less trivial solutions where the rank of the Poisson tensor is non-zero in general. Rather than discussing them in full generality we specify now the Poisson tensor such that we describe two-dimensional dilaton gravity. We use suggestive notation and denote the target space coordinates as $X^I = (X,\, X^a)$ and the connection 1-forms as $A_I = (\omega,\, e_a)$. Choosing the Poisson tensor as

$$P^{aX} = \sigma\epsilon^a{}_b X^b \qquad P^{ab} = -\epsilon^{ab}\left(V(X) + \tfrac{\sigma}{2}\, X^c X_c U(X)\right) \tag{35}$$

where we contract indices a, b with the Minkowski metric η_{ab} for $\sigma = -1$ and with the Euklidean metric δ_{ab} for $\sigma = 1$, the PSM action (28) with the choices above is equivalent to dilaton gravity in first order formulation (18).

Thus, in the same way that the Einstein–Hilbert–Palatini action is a special case of a Chern–Simons action (with additional structure that allows us to identify which part of the gauge connection is the metric), the first order action (18) is a special case of a PSM (with additional structure that allows us to identify which part of the gauge connection is the metric).

After these generalities let us start with some calculations. Our task is to find all solutions to the equations of motion for generic dilaton gravity models. If you look back at the second order equations (24) this seems like a daunting task, but actually everything is going to be rather simple. To get going let us introduce light-cone coordinates so that the Minkowski metric reads $\eta_{+-} = 1$ and the epsilon-symbol is given by $\epsilon^\pm{}_\pm = \pm 1$. The first order action (18) [or equivalently, the PSM action (28) with the Poisson tensor (35)] with $\sigma = -1$ then reads

$$I^{(1)}[X,\, X^\pm,\, \omega,\, e^\pm] = \frac{k}{2\pi}\int \Big[X^-\big(\mathrm{d}e^+ - \omega\wedge e^+\big) + X^+\big(\mathrm{d}e^- + \omega\wedge e^-\big) + X\,\mathrm{d}\omega \\ + \big(V(X) - X^+X^-U(X)\big)\, e^+\wedge e^-\Big]\,. \tag{36}$$

It will be useful to introduce the following integrals of the potentials U, V:

$$Q(X) := \int^X \mathrm{d}y\, U(y) \qquad\qquad w(X) := 2\int^X \mathrm{d}y\, V(y) e^{Q(y)} \tag{37}$$

It is sometimes convenient to transform to a different, conformally related, theory. Under dilaton-dependent Weyl rescalings, $g_{\mu\nu} \to g_{\mu\nu} e^{2\Omega(X)}$,

$$X^a \to X^a e^{-\Omega} \qquad e^a \to e^a e^{\Omega} \qquad \omega \to \omega + X^a e_a\, \Omega' \tag{38}$$

the action (36) is mapped to an action of the same type, but with new potentials

$$U \to U + 2\Omega' \qquad\qquad V \to V e^{-2\Omega}\,. \tag{39}$$

The functions defined in (37) have simple transformation behavior,

$$Q \to Q + 2\Omega \qquad\qquad w \to w\,. \tag{40}$$

The function w is invariant under all dilaton-dependent Weyl rescalings.

5.5 All classical solutions

We exploit Weyl rescalings to map a dilaton gravity model with non-vanishing U to another dilaton gravity model with vanishing U,[2] find all solutions for that model and in the end do the inverse Weyl rescaling to get all solutions of the original theory. Thus, we assume now $U = 0$.

The equations of motion for vanishing U are given by

$$\mathrm{d}X - X^- e^+ + X^+ e^- = 0 \tag{41}$$

$$\mathrm{d}X^\pm \mp X^\pm \omega \mp e^\pm\, V = 0 \tag{42}$$

$$\mathrm{d}\omega + e^+ \wedge e^-\, V' = 0 \tag{43}$$

$$\mathrm{d}e^\pm \mp \omega \wedge e^\pm = 0\,. \tag{44}$$

5.5.1 Linear dilaton vacua

Let us assume for the time being $X^+ \neq 0$ in a given patch and define a new 1-form Z by $e^+ = X^+ Z$. Then (41) implies $e^- = X^- Z - \mathrm{d}X/X^+$. The upper sign equation (42) yields the connection $\omega = \mathrm{d}X^+/X^+ - V\, Z$. The upper sign torsion constraint (44) establishes that Z is closed, $\mathrm{d}Z = 0$. Therefore, locally Z is exact.

$$Z = \mathrm{d}u \tag{45}$$

So far we have used only half of the equations of motion. You can check that two of the remaining three equations are redundant with the rest. Thus, we need to consider only one additional equation. For instance, taking the linear combination of X^- times the upper sign equation (42) plus X^+ times the lower sign equation (42) and using (41) yields

$$\mathrm{d}(X^+ X^-) - V\ \mathrm{d}X = 0\,. \tag{46}$$

Since V depends on X only we can integrate (46), calling the integration constant (minus) M.

$$X^+ X^- - \frac{1}{2}\, w(X) = -M \tag{47}$$

Using as coordinates the dilaton field X and u the metric is then given by

$$\mathrm{d}s^2_{U=0} = 2e^+ \otimes e^- = 2X^+\ \mathrm{d}u\,(X^-\ \mathrm{d}u - \mathrm{d}X/X^+) = \big(w(X) - 2M\big)\ \mathrm{d}u^2 - 2\,\mathrm{d}u\ \mathrm{d}X\,. \tag{48}$$

[2] If you are familiar with the concept of an "Einstein frame" please note that in two dimensions there is no such frame, i.e., unlike in higher dimensions it is impossible to find a Weyl rescaling that maps the term $\sqrt{-g}\,XR$ to $\sqrt{-\tilde{g}}\,\tilde{R}$. This is so, because in two dimensions the volume form has the same Weyl weight as the metric, implying Weyl-invariance of $\sqrt{-g}\,g^{\mu\nu}$.

Transforming back to the original conformal frame finally establishes the general solution of all dilaton gravity models in an Eddington–Finkelstein patch,

$$\boxed{\mathrm{d}s^2 = e^{Q(X)} \big(w(X) - 2M\big)\, \mathrm{d}u^2 - 2\,\mathrm{d}u\, \mathrm{d}r} \tag{49}$$

where we have redefined the radial coordinate as $\mathrm{d}r = e^{Q(X)}\, \mathrm{d}X$. By inspection we see that all solutions (49) have a Killing vector ∂_u, which is a generalized Birkhoff theorem (for spherically reduced gravity from four dimensions this statement is equivalent to the Birkhoff theorem). The norm of this Killing vector vanishes when

$$\text{Killing horizons:} \qquad w(X) = 2M\,. \tag{50}$$

Since the corresponding $r =$ const. hypersurface is null, we have a Killing horizon for each such zero.

The Ricci-scalar following from the metric (49) is given by

$$R_{\text{LDV}} = 2e^{-Q(X)}\, \frac{\mathrm{d}}{\mathrm{d}X} \Big(e^{Q(X)} V(X) + U(X) \big(M - \tfrac{1}{2}\, w(X)\big)\Big) \tag{51}$$

where "LDV" stands for "linear dilaton vacuum". Models with vanishing (constant positive) [constant negative] Ricci scalar for $M = 0$ are called Minkowski/Rindler (dS) [AdS] ground state models. The discrimination between Minkowski and Rindler works as follows: if additionally to vanishing Ricci scalar for $M = 0$ we have the relation $e^Q w =$ const. then the model has a Minkowski ground state, otherwise a Rindler ground state.

The Poisson-tensor associated with the generic solutions (49) has rank 2 and thus a 1-dimensional kernel. The 1-dimensional kernel implies the existence of a Casimir function, given by the left hand side of (47), which is constant on-shell, $\partial_u M = \partial_r M = 0$. This observation implies the existence of a conserved mass denoted by M.

5.5.2 Constant dilaton vacua

Our solutions cease to be valid when $X^+ = 0$. Therefore, we investigate now what happens when $X^+ = 0$. There are two possibilities: either $X^+ = 0$ everywhere or $X^+ = 0$ on some co-dimension 1 hypersurface. In the first case the upper sign equation (42) implies that X must be constant and a solution of $V(X) = 0$. Thus, this case reduces to a constant dilaton vacuum. Equation (41) with $X^+ = 0$ and constant X yields $X^- = 0$ so that the Poisson tensor vanishes identically. Equation (43) then establishes that constant dilaton vacua have to be constant curvature solutions, with Ricci scalar given by

$$R_{\text{CDV}} = 2\frac{\mathrm{d}V}{\mathrm{d}X}\Big|_{X=X_0} \qquad\qquad V(X_0) = 0 \qquad\qquad X^{\pm} = 0\,. \tag{52}$$

5.5.3 Beyond basic Eddington–Finkelstein patches

In the second case, where $X^+ = 0$ not in an open region but only on a codimension-1 hypersurface, we can repeat the analysis of linear dilaton vacua starting with the assumption $X^- \neq 0$ instead of $X^+ \neq 0$, mutatis mutandis. This will lead to linear dilaton vacua valid in mirror-flipped Eddington–Finkelstein patches, swapping the role of ingoing and outgoing horizons.

The only point that we cannot cover in this way is the bifurcation point $X^+ = X^- = 0$, which fits well with our considerations about Eddingston–Finkelstein patches and Penrose diagrams in Black Holes II. While they are not difficult to treat, we shall ignore bifurcation points in these lectures.

5.5.4 Example: the ab-family

An interesting two-parameter family of models is given by monomial potentials

$$U(X)=-\frac{a}{X}\qquad\qquad V(X)\propto X^{a+b} \tag{53}$$

which (after a suitable choice of the proportionality constant and integration constant in the definition of Q) leads to

$$e^{Q(X)}=X^{-a}\qquad\qquad w(X)=X^{b+1}\,. \tag{54}$$

The family of models defined by (53) comprises spherically reduced gravity from any dimension ($a=(D-3)/(D-2)$, $b=-1/(D-2)$), the Witten black hole ($a=1$, $b=0$), the JT model ($a=0$, $b=1$) and numerous other dilaton gravity models with different asymptotics and Penrose diagrams. The Schwarzschild black hole is the special case $a=1/2=-b$.

Minkowski ground state models obey the linear relation $a=1+b$, while (A)dS ground state models obey $a=1-b$ (whether the ground state is dS or AdS depends on the sign of the proportionality constant in (53); if it is negative we have AdS). Rindler ground state models obey $b=0$. A "phase diagram" of Penrose diagrams for all values of a,b is displayed in Fig. 3.12 of hep-th/0204253. Thermodynamic properties of the ab-family are discussed in section 4.2 of hep-th/0703230.

More on global properties of all classical solutions to two-dimensional dilaton gravity can be found in a series of papers by Klösch and Strobl.

5.6 Analogue of Brown–Henneaux boundary conditions?

Inspired by the gauge theoretic results in three dimensions we can propose Brown–Henneaux like boundary conditions for the JT model, where the Poisson-σ model reduces to a non-abelian BF theory with an $sl(2,\mathbb{R})$ connection. Inspired guessing leads to the ansatz

$$A=b^{-1}\big(\,\mathrm{d}{+}a\big)\,b\qquad\qquad X=b^{-1}xb \tag{55}$$

with the group element $b=e^{\rho\,L_0}$ and the connection 1-form

$$a=\big(L_{+1}-\mathcal{L}(t)\,L_{-1}\big)\;\mathrm{d}t \tag{56}$$

where $L_{\pm1},L_0$ are again the usual $sl(2,\mathbb{R})$ generators. The scalar field x is hard to guess, but actually can be constructed with the data above — see section 3 in 1802.01562.

In the metric formulation these boundary conditions were found in section 6.2 of 1708.08471.

$$\mathrm{d}s^2=\mathrm{d}\rho^2-\big(\tfrac14\,e^{2\rho}+\mathcal{O}(1)\big)\;\mathrm{d}t^2\qquad\qquad X=\mathcal{O}(e^{\rho})+\mathcal{O}(e^{-\rho}) \tag{57}$$

A surprising aspect of these boundary conditions is that the dilaton is allowed to fluctuate to leading order, a possibility considered first in 1311.7413 and 1509.08486.

Such boundary conditions turn out to be useful for the holographic correspondence to the SYK model, see 1604.07818, 1606.01857 and references therein. We shall come back to the SYK model later. In the next section we turn to gravity/gauge theory tools of relevance for holography, namely the construction of canonical boundary charges.