2 Asymptotic Killing vectors

Killing vectors generate isometries of metrics and preserve a given metric, in the sense that the Lie variation along a Killing vector vanishes when acting on the metric (see section 6.11 in Black Holes I). Killing vectors lead to conserved charges, defined by Komar integrals (see section 8.4 in Black Holes I). The main purpose of this section is to generalize this notion to situations where the metric is not necessarily preserved, but only its asymptotic structure. We discuss general aspects of asymptotic Killing vectors and consider also a few examples.

2.1 Definition of asymptotic Killing vectors

Asymptotic Killing vectors do not necessarily preserve any given metric, but they preserve the asymptotic structure, so when acting with a Lie derivative along an asymptotic Killing vector on a metric that obeys some specifed set of boundary conditions the result is not necessarily zero, but rather can be a fluctuation allowed by these boundary conditions. The defining equation for asymptotic Killing vectors

$$\boxed{\mathcal{L}_\xi g_{\mu\nu} = \mathcal{O}\big(\delta g_{\mu\nu}\big)} \tag{1}$$

has on the left hand side the Lie derivative along the asymptotic Killing vector ξ of a metric g compatible with some given set of boundary conditions, and on the right hand side some metric fluctuation allowed by the boundary conditions.

Once the asymptotic Killing vectors are determined it is also of interest to check their Lie brackets, since this will eventually lead to the asymptotic symmetry algebra discussed in the next subsection.

In many practical applications one introduces a (partial) gauge fixing and then requires additionally that the fluctuations δg also preserve the gauge conditions, for instance a gauge fixing to Gaussian normal coordinates.

Example. Let us consider a baby-example to make the notion of asymptotic Killing vectors in presence of partial gauge fixing more concrete. Take the two-dimensional class of metrics defined near the asymptotic boundary at $r \to \infty$ by

$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \sum_{n=-1}^{\infty} g_n(u) r^{-n}\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r \tag{2}$$

where all coefficients g_n are allowed to vary so that the allowed metric variations are given by

$$\delta g_{uu} = \mathcal{O}(r) + \mathcal{O}(1) + \mathcal{O}(1/r) + \dots \qquad\qquad \delta g_{ur} = \delta g_{rr} = 0\,. \tag{3}$$

The allowed fluctuations (3) preserve Eddington–Finkelstein gauge to all orders in the radial coordinate r (a condition which could be relaxed without changing the physics) and lead to asymptotically flat metrics, in the sense that the Ricci-scalar tends to zero as r tends to infinity (at least like $1/r^3$). Let us now construct the associated asymptotic Killing vectors. The rr-component of (1) yields

$$\xi^\mu\partial_\mu g_{rr} + 2g_{r\mu}\partial_r\xi^\mu = -2\partial_r\xi^u = 0 \tag{4}$$

from which we conclude that the u-component of ξ is r-independent. The ru-component of (1) is more complicated,

$$\xi^\mu\partial_\mu g_{ru} + g_{r\mu}\partial_u\xi^\mu + g_{u\mu}\partial_r\xi^\mu = -\partial_u\xi^u - \partial_r\xi^r + g_{uu}\partial_r\xi^u = 0 \tag{5}$$

and together with the previous condition (4) allows to deduce the following results for the asymptotic Killing vectors,

$$\xi = \epsilon(u)\,\partial_u + \big(-\epsilon'(u)\,r + \eta(u)\big)\,\partial_r \tag{6}$$

where prime denotes derivative with respect to u. The uu-component of (1) does not lead to any new constraint on the asymptotic Killing vectors,

$$\xi^\mu \partial_\mu g_{uu} + 2g_{u\mu}\partial_u \xi^\mu = \mathcal{O}(r) + \mathcal{O}(1) + \mathcal{O}(1/r) + \dots\,. \tag{7}$$

Since the asymptotic Killing vectors (6) of the metric (2) are parametrized by two arbitrary functions, $\epsilon(u)$ and $\eta(u)$, we have infinitely many asymptotic Killing vectors for this example. As we shall see in the next subsection it is of interest to calculate the algebra of Lie-brackets between the asymptotic Killing vectors; for the present examples we obtain after a small calculation

$$\big[\xi(\epsilon_1,\eta_1),\,\xi(\epsilon_2,\eta_2)\big]_{\text{Lie}} = \xi\big(\epsilon_1\epsilon_2' - \epsilon_2\epsilon_1',\,(\epsilon_1\eta_2 - \epsilon_2\eta_1)'\big)\,. \tag{8}$$

The result (8) shows that we have a linear algebra, the Lie-bracket algebra of the asymptotic Killing vectors. This algebra generates the asymptotic symmetries for our example.

2.2 Asymptotic symmetries

Noether's theorem relates the conservation of the asymptotic structure of the metric (and possibly other fields that might be present) to a set of symmetries known as asymptotic symmetries.

It seems tempting to define asymptotic symmetries as the set of all transformations generated by the asymptotic Killing vectors. However, we have a lot of (gauge) redundancy, and not every asymptotic Killing vector generates an interesting symmetry near the boundary; instead, some of the asymptotic Killing vectors may be just pure gauge transformations that remain sufficiently small at the boundary so that they have no physical effect other than changing the coordinate system. We are going to be precise about what "sufficiently small" means in later sections. For now we simply assume that there is a well-defined notion of "sufficiently small" so that the asymptotic Killing vectors can be classified into "proper gauge transformations" (those which remain sufficiently small; they are also called "trivial gauge transformations") and "improper gauge transformations" (sometimes also called "large gauge transformations; they are not pure gauge at the boundary).

The asymptotic symmetry algebra is the set of all boundary-condition preserving transformations modulo trivial gauge transformations. The asymptotic symmetry group is the group associated with the asymptotic symmetry algebra. The asymptotic symmetries are generated by elements of the asymptotic symmetry group.

Back to the example. Let us reconsider the baby-example above, but now take instead of (2) the ansatz

$$\mathrm{d}s^2 = \sum_{n=-1}^{\infty} g_n(u) r^{-n}\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r\,\Big(1 + \sum_{n=1}^{\infty} f_n(u) r^{-n}\Big) + \sum_{n=1}^{\infty} h_n(u) r^{-n}\,\mathrm{d}r^2 \tag{9}$$

where all functions g_n, f_n, h_n are allowed to fluctuate. The Ricci scalar still vanishes as r tends to infinity (albeit only with $1/r$), so in this sense the metric is still asymptotically flat. However, instead of the result (6) we find an infinitely larger set of asymptotic Killing vectors,

$$\xi = \big(\epsilon(u) + \mathcal{O}(1/r)\big)\,\partial_u + \big(-\epsilon'(u)\,r + \eta(u) + \mathcal{O}(1/r)\big)\,\partial_r\,. \tag{10}$$

Since all we did in comparison to (2) was to drop the assumption of strict gauge-fixing to Eddington–Finkelstein gauge, it is plausible that the additional Killing vectors contained in (10), i.e., the terms of order $\mathcal{O}(1/r)$, generate trivial gauge transformations associated with coordinate change to Eddington–Finkelstein. Applying the definition above the asymptotic symmetries are then generated by all Killing vectors (10) where the $\mathcal{O}(1/r)$ part is modded out — this yields precisely the set of asymptotic Killing vectors (6). Thus, we expect for this example that the asymptotic symmetries are generated by (6) and hence the asymptotic symmetry algebra is given by (8).

2.3 Asymptotically AdS$_3$ as simple example

Let us investigate the asymptotic symmetries of AdS$_3$, which is a slightly more physical example than the baby-example considered above. In fact, we studied this scenario already in section 11.4 of the lecture notes for Black Holes II. We found there that the boundary (plus gauge) conditions

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \Big(e^{2\rho/\ell}\,\gamma^{(0)}_{\mu\nu} + \gamma^{(2)}_{\mu\nu} + \dots\Big)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \qquad\qquad \delta\gamma^{(0)}_{\mu\nu} = 0 \tag{11}$$

are preserved by the asymptotic Killing vectors

$$\xi = \varepsilon^+(x^+)\partial_+ + \varepsilon^-(x^-)\partial_- - \frac{\ell}{2}\,\big(\partial_+\varepsilon^+ + \partial_-\varepsilon^-\big)\partial_\rho + \mathcal{O}\big(e^{-2\rho/\ell}\big) \tag{12}$$

where $x^\mu = \{x^+,\, x^-\}$. As we shall see later, the asymptotic symmetries are again generated by the $\mathcal{O}(1)$ part of the asymptotic Killing vectors, whereas the subleading terms generate trivial gauge symmetries. One consistency check you can perform is to slightly relax our gauge-fixing to Gaussian normal coordinates in (11) by allowing fluctuations $g_{\rho\rho} \sim \mathcal{O}(e^{-2\rho/\ell})$ and $g_{\rho\pm} \sim \mathcal{O}(e^{-2\rho/\ell})$. You should find the samy asymptotic Killing vectors as given below, up to subleading corrections.

In this example the asymptotic symmetry algebra consists of two copies of the Witt algebra

$$[\xi(\varepsilon_1^\pm),\, \xi(\varepsilon_2^\pm)]_{\text{Lie}} = \xi\big(\varepsilon_1^\pm\varepsilon_2^{\pm\prime} - \varepsilon_2^\pm\varepsilon_1^{\pm\prime}\big)\,. \tag{13}$$

If the theory is defined such that $x^\pm \sim x^\pm + 2\pi$ then introducing Fourier modes $\varepsilon_n^\pm = ie^{inx^\pm}\partial_\pm$ brings the Witt algebra (13) into a rather useful form,

$$[\varepsilon_n^\pm,\, \varepsilon_m^\pm] = (n-m)\,\varepsilon^\pm_{n+m}\,. \tag{14}$$

From a purely algebraic perspective (and, as we shall se later, also for physical reasons) it is of interest to ask whether a given asymptotic symmetry algebra has a non-trivial central extension. In the present case the answer is yes, and the centrally extended version of the Witt algebra (14) is known as Virasoro algebra with central charge c.

$$[L_n^\pm,\, L_m^\pm] = (n-m)\,L^\pm_{n+m} + \frac{c}{12}\,\big(n^3-n\big)\,\delta_{n+m,\,0}\,. \tag{15}$$

We shall see later that such central charges can arise when realizing canonically an asymptotic symmetry algebra. The central extension from Witt to Virasoro does indeed arise in three-dimensional gravity. The interpretation of the canonical realization of the asymptotic symmetries is that the physical phase space (or in the quantum theory the physical Hilbert space) falls into representations of two copies of the Virasoro algebra. On the other hand, the defining property of a conformal field theory (CFT) in two dimensions is that the physical phase space (or in the quantum theory the physical Hilbert space) falls into representations of the two-dimensional conformal algebra — which is precisely two copies of the Virasoro algebra. Thus, this examples provide a glimpse into AdS/CFT and suggests that quantum gravity on AdS$_3$ (if it exists) is equivalent to some CFT$_2$.

2.4 Asymptotically flat space as complicated example

You may wonder why we did not consider as baby-example (asymptotically) flat space, since a natural guess is that the asymptotic symmetries of asymptotically flat space are equivalent to the symmetries of Minkowski space and that overall asymptotically flat space should provide the simplest example. However, depending on the precise boundary conditions there can be a much larger set of asymptotic symmetries than just the one given by the Poincaré group, and asymptotically flat space can actually be intriguingly complicated.

Following the spirit of these lectures, let us consider asymptotically flat space in three spacetime dimensions. Due to the null structure of asymptotic infinity in Minkowski space ($\mathscr{I}^\pm$ are null hypersurfaces) it is convenient to focus on one half of the asymptotic boundary (e.g. $\mathscr{I}^+$) and formulate the boundary conditions in a gauge adapted to this situation, e.g. Eddington–Finkelstein or Bondi coordinates.

$$\begin{aligned} \mathrm{d}s^2 = \big(h_{uu} + \mathcal{O}(1/r)\big)\, \mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r\,\big(1+\mathcal{O}(1/r)\big) + \big(h_{u\varphi} + \mathcal{O}(1/r)\big)\,\mathrm{d}u\,\mathrm{d}\varphi \\ + r^2\,\mathrm{d}\varphi^2\,\big(1+\mathcal{O}(1/r)\big) \end{aligned} \tag{16}$$

With some anticipation we fixed $g_{rr} = g_{r\varphi} = 0$ by exploiting small diffeomorphisms; it is not necessary to do this, but it makes our calculations shorter which is why we made the choice (16). The functions h_{uu} and $h_{u\varphi}$ are allowed to vary and to depend on retarded time u and angular coordinate $\varphi \sim \varphi + 2\pi$, but not on the radial coordinate r. You can convince yourself that the metric (16) is asymptotically flat in the sense that the independent polynomial curvature invariants behave as $R \sim \mathcal{O}(1/r^2)$, $R_{\mu\nu}R^{\mu\nu} \sim \mathcal{O}(1/r^4)$, and $R_{\mu\nu}R^{\nu\lambda}R^\mu_\lambda \sim \mathcal{O}(1/r^6)$.

The boundary (plus gauge) conditions (16) are preserved by the asymptotic Killing vectors

$$\xi = \big(M(\varphi) + uL'(\varphi)\big)\,\partial_u + \Big(L(\varphi) - \frac{u}{r}\,L''(\varphi) - \frac{1}{r}M'(\varphi)\Big)\,\partial_\varphi - \big(rL'(\varphi) + \mathcal{O}(1/r)\big)\,\partial_r\,. \tag{17}$$

The results above look quite complicated, but simplify if we restrict to leading order in a large-r expansion and split the asymptotic Killing vectors (17) into L- and M-dependent pieces,

$$\xi^L = uL'(\varphi)\,\partial_u + \Big(L(\varphi) - \frac{u}{r}\,L''(\varphi)\Big)\,\partial_\varphi - \big(rL'(\varphi) + \mathcal{O}(1/r)\big)\,\partial_r \tag{18}$$

$$\xi^M = M(\varphi)\,\partial_u + \mathcal{O}(1/r)\,. \tag{19}$$

Their Lie-bracket algebra has again infinitely many generators and reads

$$[\xi^L(L_1),\,\xi^L(L_2)]_{\text{Lie}} = \xi^L\big(L_1L_2' - L_2L_1'\big) + \mathcal{O}(1/r) \tag{20}$$

$$[\xi^L(L_1),\,\xi^M(M_2)]_{\text{Lie}} = \xi^M\big(L_1M_2' - M_2L_1'\big) + \mathcal{O}(1/r) \tag{21}$$

$$[\xi^M(M_1),\,\xi^M(M_2)]_{\text{Lie}} = \mathcal{O}(1/r)\,. \tag{22}$$

Comparing the first line (20) with (13) we see that the Witt algebra is recovered as subalgebra. Geometrically, this makes sense since to leading order the asymptotic Killing vector ξ^L generates diffeomorphisms of the celestial S^1. The last line (22) shows that the asymptotic Killing vectors ξ^M commute with each other to leading order. Since their zero mode, $\xi^M_0 = \partial_u$, generates time-translations (which are part of Poincaré) the asymptotic Killing vectors ξ^M are known as "supertranslations" (note: no relation to supersymmetry). To have suggestive names, sometimes the Witt algebra generators ξ^L are called "superrotations", as the zero mode $\xi^L_0 = \partial_\varphi$ generates rotations. Superrotations and supertranslations do not commute (21), but instead yield something reminiscent of a Witt algebra.

All central extensions of the algebra (20)-(22) are known and will be discussed once we address aspects of flat space holography in three bulk dimensions.