1 Summary of gravity with boundaries

On this sheet of paper we give a brief reminder of essential aspects related to gravity in the presence of boundaries. For sake of concreteness we assume that the boundary is timelike, which is true e.g. for asymptotically AdS boundaries or stretched horizons. If you are completely unfamiliar with gravity in the presence of boundaries you may wish to consult my lecture notes for the course Black Holes II, section 10.

1.1 Canonical decomposition of the metric

The canonical decomposition of a D-dimensional metric $g_{\mu\nu}$ (often referred to as "bulk metric") into a $(D-1)$-dimensional metric $h_{\mu\nu}$ (often referred to as "boundary metric", "induced metric" or "first fundamental form"), and a normal vector n^μ normalized to unity, $n^\mu n_\mu = +1$, reads

$$g_{\mu\nu} = h_{\mu\nu} + n_\mu n_\nu \,. \tag{1}$$

Recall that the boundary metric is still a D-dimensional symmetric tensor, but projects out the normal component,

$$h_{\mu\nu} n^\nu = 0 \qquad\qquad h^\mu_\mu = D - 1 \,. \tag{2}$$

The projected velocity with which the normal vector changes (often referred to as "extrinsic curvature" or "second fundamental form"),

$$K_{\mu\nu} = h^\alpha_\mu h^\beta_\nu \nabla_\alpha n_\beta = \frac{1}{2} \big(\mathcal{L}_n h\big)_{\mu\nu} \tag{3}$$

also is a symmetric tensor and has vanishing contraction with the normal vector,

$$K_{\mu\nu} = K_{\nu\mu} \qquad\qquad K_{\mu\nu} n^\mu = 0 \,. \tag{4}$$

The trace of extrinsic curvature is denoted by K,

$$K = K^\mu_\mu = \nabla_\mu n^\mu \,. \tag{5}$$

Projection with the boundary metric yields a boundary-covariant derivative

$$\mathcal{D}_\mu = h^\nu_\mu \nabla_\nu \tag{6}$$

that leads to standard (pseudo-)Riemann tensor calculus at the boundary when acting on tensors projected to the boundary.

1.2 Gaussian normal coordinates

Sometimes it is convenient to introduce Gaussian normal coordinates when discussing boundaries. Let us assume that the boundary can be characterized as a $\rho =$ const. hypersurface, where ρ is one of the coordinates. In so-called ADM variables the metric reads [i, j run from 0 to $(D-2)$]

$$g_{\mu\nu}\, \mathrm{d}x^\mu \, \mathrm{d}x^\nu = N \, \mathrm{d}\rho^2 + \gamma_{ij} \left(\mathrm{d}x^i + N^i \, \mathrm{d}\rho\right) \left(\mathrm{d}x^j + N^j \, \mathrm{d}\rho\right) \tag{7}$$

where the "lapse function" N and the "shift-vector" N^i are functions of all coordinates. Gaussian normal coordinates mean that one chooses a gauge where the lapse function is set to unity and the shift vector to zero. Thus, in Gaussian normal coordinates with respect to the coordinate ρ the metric simplifies to

$$g^{\text{GNC}}_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu = \mathrm{d}\rho^2 + \gamma_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j \,. \tag{8}$$

Note that the quantity γ_{ij} is nothing but the boundary metric, which obeys $h_{ij} = \gamma_{ij}$ and $h_{i\rho} = h_{\rho\rho} = 0$.

One advantage of Gaussian normal coordinates is that the normal vector is rather simple, $n^\rho = n_\rho = 1$, $n^i = n_i = 0$, implying that extrinsic curvature can be calculated quickly by hand,

$$K_{ij}^{\text{GNC}} = \frac{1}{2}\,\partial_\rho\gamma_{ij} \qquad\qquad K_{i\rho}^{\text{GNC}} = K_{\rho\rho}^{\text{GNC}} = 0\,. \tag{9}$$

1.3 Variation of Einstein–Hilbert action with boundary terms

The full action for Einstein gravity (compatible with a Dirichlet boundary value problem consists of the bulk action I_{EH} plus a boundary action I_{GHY}, known as Gibbons–Hawking–York boundary term.

$$I = I_{\text{EH}} + I_{\text{GHY}} = \frac{1}{16\pi G}\int_{\mathcal{M}}\mathrm{d}^D x\sqrt{-g}\big(R - 2\Lambda\big) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}}\mathrm{d}^{D-1}x\sqrt{-h}\,K \tag{10}$$

Its first variation (assuming a smooth boundary) is given by

$$\begin{aligned}\delta I = -\frac{1}{16\pi G}\int_{\mathcal{M}}\mathrm{d}^D x\sqrt{-g}\,\Big(R^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}R + \Lambda g^{\mu\nu}\Big)\,\delta g_{\mu\nu} \\ - \frac{1}{16\pi G}\int_{\partial\mathcal{M}}\mathrm{d}^{D-1}x\sqrt{-h}\,\big(K^{\mu\nu} - h^{\mu\nu}K\big)\,\delta g_{\mu\nu}\end{aligned} \tag{11}$$

The tensor multiplying the variation $\delta g_{\mu\nu}$ at the boundary is known as **Brown–York stress tensor**,

$$T_{\text{BY}}^{\mu\nu} := \frac{1}{8\pi G}\,\big(K^{\mu\nu} - h^{\mu\nu}K\big)\,. \tag{12}$$

It is important to recall that further boundary terms can be added to the action (10) without spoiling the Dirichlet boundary value problem, for instance by adding further boundary terms that depend only on curvature invariants constructed from the boundary metric. As you should know already, these terms are actually necessary in many applications. The reason for this is that even though we have a well-defined Dirichlet boundary value problem we still may not have a well-defined action principle, in the sense that there could be allowed variations of the metric that do not lead to a vanishing first variation (11) on some solutions of the equations of motion.

An example that we discussed in section 11 of Black Holes II is AdS_3 gravity with Brown–Henneaux boundary conditions,

$$\mathrm{d}s^2\big|_{\text{aAdS}} = \mathrm{d}\rho^2 + \Big(e^{2\rho/\ell}\,\gamma_{\mu\nu}^{(0)}(x^\alpha) + \gamma_{\mu\nu}^{(2)}(x^\alpha) + \dots\Big)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{13}$$

with variations

$$\delta\gamma_{\mu\nu}^{(0)} = 0 \qquad\qquad \delta\gamma_{\mu\nu}^{(2)} \neq 0\,. \tag{14}$$

The full action compatible with these boundary conditions is given by

$$\Gamma_{\text{AdS}_3} = \frac{1}{16\pi G}\int_{\mathcal{M}}\mathrm{d}^3x\sqrt{-g}\,\Big(R + \frac{2}{\ell^2}\Big) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}}\mathrm{d}^2x\sqrt{-h}\,\Big(K - \frac{1}{\ell}\Big) \tag{15}$$

which leads to a finite ("holographically renormalized") Brown–York stress tensor.

$$T_{\mu\nu}^{\text{BY-ren}} = \frac{1}{8\pi G}\,\Big(K_{\mu\nu} - h_{\mu\nu}K + h_{\mu\nu}\,\frac{1}{\ell}\Big) = -\frac{1}{8\pi G\,\ell}\,\gamma_{\mu\nu}^{(2)} \tag{16}$$