9 Hawking effect

Black holes at finite surface gravity κ emit radiation that to leading order approximation is thermal. This is known as Hawking effect. The purpose of this section is to calculate the Hawking temperature in terms of surface gravity, i.e., to determine the precise $\mathcal{O}(1)$ coefficient in the relation $\kappa \sim T$. We shall do this in two different ways, by Euclidean continuation and by a semi-classical calculation of scalar field fluctuations on a black hole background.

9.1 Periodicity in Euclidean time is inverse temperature

Quantum mechanically unitary time evolution of some state $|\psi(0)\rangle$ is generated by some Hermitean Hamiltonian H,

$$|\psi(t)\rangle = e^{iHt}|\psi(0)\rangle\,. \tag{1}$$

Quantum statistically, the partition function is defined by a trace over the Boltzmann factor $e^{-\beta H}$, where $\beta = T^{-1}$ is inverse temperature,

$$Z = \mathrm{tr}\big(e^{-\beta H}\big) = \sum_\psi \langle\psi(0)|e^{-\beta H}|\psi(0)\rangle = \sum_\psi e^{-\beta E_\psi} \tag{2}$$

where the sum is over a complete set of states $|\psi(0)\rangle$ and E_ψ are energy eigenvalues. The key observation here is that the Boltzmann factor can be reinterpreted as time evolution of the state $|\psi(0)\rangle$ over the imaginary time period $-i\beta$, thus yielding

$$Z = \sum_\psi \langle\psi(0)|\psi(-i\beta)\rangle = \sum_\psi \langle\psi(+i\beta)|\psi(0)\rangle\,. \tag{3}$$

Given the expressions (3) for the partition function it is suggestive to impose periodicity in the imaginary part of time,

$$t \sim t - i\beta \quad \Rightarrow \quad \tau \sim \tau + \beta \quad \text{where } \tau = it\,. \tag{4}$$

Periodicity in Euclidean time τ is identical to inverse temperature β.

Actually, for those who know a bit of QFT let us be more concrete and consider the Green function of a free theory at finite temperature,

$$G(x-y) = \frac{\sum_\psi \langle\psi|T(\phi(x)\phi(y))|\psi\rangle e^{-\beta E_\psi}}{\sum_\psi e^{-\beta E_\psi}} = \frac{1}{Z}\,\mathrm{tr}\big(e^{-\beta H}T(\phi(x)\phi(y))\big) \tag{5}$$

where the $|\psi\rangle$ are eigenstates of H with eigenvalues E_ψ and T denotes time-ordering. We then get the following chain of identities (assuming $x^0 > 0$ we can drop time ordering in the first step)

$$\begin{aligned} G(x^0,\vec{x};0,\vec{y}) &= \frac{1}{Z}\,\mathrm{tr}\big(e^{-\beta H}\phi(x^0,\vec{x})\phi(0,\vec{y})\big) = \frac{1}{Z}\,\mathrm{tr}\big(\phi(0,\vec{y})e^{-\beta H}\phi(x^0,\vec{x})\big) \\ &= \frac{1}{Z}\,\mathrm{tr}\big(e^{-\beta H}e^{\beta H}\phi(0,\vec{y})e^{-\beta H}\phi(x^0,\vec{x})\big) = \frac{1}{Z}\,\mathrm{tr}\big(e^{-\beta H}\phi(-i\beta,\vec{y})\phi(x^0,\vec{x})\big) \\ &= \frac{1}{Z}\,\mathrm{tr}\big(e^{-\beta H}T(\phi(x^0,\vec{x})\phi(-i\beta,\vec{y}))\big) = G(x^0,\vec{x};-i\beta,\vec{y}) \end{aligned} \tag{6}$$

Perhaps the least obvious step is the penultimate equality, where we applied time-ordering in presence of imaginary time. Comparing the initial and the final expressions shows periodicity of the finite temperature Green function in Euclidean time with period $\beta = T^{-1}$. Thus, in a quantum field theory the defining signature of a thermal state at temperature T is periodicity in Euclidean time, a conclusion we also reached above. This is also known as KMS condition.

Thus, if you construct a physical state and can show that it has to be periodic in Euclidean time τ with period β, i.e., $\tau \sim \tau + \beta$, you can deduce it is a thermal state at temperature $T = 1/\beta$.

9.2 Hawking temperature from Euclidean regularity

Consider now a D-dimensional spacetime with a non-extremal Killing horizon with surface gravity $\kappa > 0$. As we have shown in the last semester, near the horizon we can universally approximate the spacetime as two-dimensional Rindler spacetime together with some transversal space,

$$\mathrm{d}s^2 = -\kappa^2 r^2\, \mathrm{d}t^2 + \mathrm{d}r^2 + g_{ij}^{\text{trans}}\, \mathrm{d}x^i\, \mathrm{d}x^j \tag{7}$$

where $i, j = 2, 3, \dots, D$. For instance, for Schwarzschild $g_{ij}^{\text{trans}}\, \mathrm{d}x^i\, \mathrm{d}x^j$ is the metric of the round two-sphere. Continuing (7) to Euclidean signature, $\tau = it$, yields

$$\mathrm{d}s^2 = r^2\, \mathrm{d}(\kappa\tau)^2 + \mathrm{d}r^2 + \dots \tag{8}$$

where we displayed only the (Euclidean) Rindler part of the metric (see also exercise 9.3). They key observation is that the space defined by the metric (8) locally is just flat Euclidean space in polar coordinates. Globally, however, the metric in general has a conical singularity at $r \to 0$. The only way to avoid this singularity is to make $\kappa\tau$ periodic with period 2π.

We have just derived that regularity of a Killing horizon in Euclidean signature implies Euclidean time is periodic with period $2\pi/\kappa$. Thus, given the considerations of the previous subsection we arrive at an important conclusion. **Spacetimes with a Killing horizon at surface gravity $\boldsymbol{\kappa > 0}$ are thermal states with Hawking–Unruh temperature**

$$\boxed{T = \frac{\kappa}{2\pi}} \tag{9}$$

Note that this conclusion applies to all types of Killing horizons, including event horizons of stationary black holes, cosmological horizons and acceleration horizons.

An important consequence of (9) is that together with the four laws it fixes the numerical factor in the **Bekenstein–Hawking entropy law**

$$\boxed{S_{\text{BH}} = \frac{A_{\text{horizon}}}{4}\,.} \tag{10}$$

9.3 Semi-classical aspects of Hawking radiation

This subsection is again directed towards students familiar with basic aspects of QFT. As in our discussion of black hole perturbations consider a scalar field ϕ on a fixed (black hole) background. Since the Klein–Gordon equation is second order in derivatives we obtained two linearly independent solutions (for each value of the angular quantum number l), so in total the solution was

$$\phi(x) = \sum_i \big(a_i \psi_i(x) + a_i^* \psi_i^*(x)\big) \tag{11}$$

where the sum extends over a complete basis, a_i denote the amplitudes and $\psi_i(x)$ are solutions to the Klein–Gordon equation on a black hole background.

In QFT the amplitudes are replaced by creation and annihilation operators,

$$\phi(x) = \sum_i \big(a_i \psi_i(x) + a_i^\dagger \psi_i^*(x)\big) \tag{12}$$

obeying the Heisenberg algebra (all commutators not displayed vanish)

$$[a_i,\, a_j^\dagger] = \delta_{ij}\,. \tag{13}$$

The QFT Hilbert space is the usual Fock space that starts from a vacuum $|0\rangle$ defined by the conditions

$$a_i|0\rangle = 0 \qquad \forall i \tag{14}$$

together with normalization $\langle 0|0\rangle = 1$. Non-vacuum states in this Fock space are generated by acting on the vacuum with creation operators $a_i^\dagger$.

Let us now choose a different basis of solutions, $\tilde{\psi}_i$, defined by

$$\tilde{\psi}_i = \sum_j \left(A_{ij}\psi_j + B_{ij}\psi_j^*\right) \tag{15}$$

subject to the normalization conditions[1]

$$A^\dagger A - B^\dagger B = \mathbb{1} \qquad\qquad AB^T = BA^T\,. \tag{16}$$

The resulting annihilation operators also transform correspondingly,

$$\tilde{a}_i = \sum_j \left(a_j A_{ji} + a_j^\dagger B_{ji}^*\right) \tag{17}$$

Such a change of basis is known as Bogoliubov-transformation with Bogoliubov coefficients A_{ij} and B_{ij}. Note that for $B_{ij} = 0$ this basis change preserves the vacuum, in the sense that the conditions (14) are identical to the similar conditions with a_i replaced by $\tilde{a}_i$. However, this is no longer true when $B_{ij} \neq 0$! A consequence of this is that **the original vacuum becomes an excited state with respect to the new basis.**

To show this important statement more explicitly consider the number operator for the i^{th} mode in the original basis,

$$N_i = a_i^\dagger a_i \tag{18}$$

and consider its expectation value in the original vacuum,

$$\langle 0|N_i|0\rangle = \langle 0|a_i^\dagger a_i|0\rangle = 0 \tag{19}$$

which vanishes. Now take instead the number operator for the i^{th} mode in the new basis

$$\tilde{N}_i = \tilde{a}_i^\dagger \tilde{a}_i = \sum_j \left(a_j^\dagger A_{ji}^* + a_j B_{ji}\right)\sum_k \left(a_k A_{ki} + a_k^\dagger B_{ki}^*\right) \tag{20}$$

and consider its expectation value in the original vacuum (in the new vacuum it vanishes by construction),

$$\begin{aligned}\langle 0|\tilde{N}_i|0\rangle &= \sum_{j,k}\langle 0|a_j B_{ji} a_k^\dagger B_{ki}^*|0\rangle = \sum_{j,k}\langle 0|a_j a_k^\dagger|0\rangle B_{ji}B_{ki}^* \\ &= \sum_{j,k}\langle 0|[a_j,\, a_k^\dagger]|0\rangle B_{ji}B_{ki}^* = \sum_j B_{ji}B_{ij}^\dagger = \left(B^\dagger B\right)_{ii} \neq 0 \end{aligned} \tag{21}$$

Let us now apply Bogoliubov transformations to a scalar field propagating on a black hole background. The key observation is that a mode that has positive frequency at late times (near $\mathscr{I}^+$)

$$\psi_\omega \sim e^{-i\omega(t-r_*)} \tag{22}$$

in general is a mixture of positive and negative frequency modes at early times (near $\mathscr{I}^-$). Similarly, positive frequency modes near $\mathscr{I}^-$ form a mixture of positive and negative frequency modes near $\mathscr{I}^+$. We saw this explicitly when discussing solutions to the Regge–Wheeler equation a few lectures ago. In terms of Bogoliubov

[1] These conditions leave invariant the symplectic inner product $\langle\psi_i,\psi_j\rangle = \delta_{ij} = -\langle\psi_i^*,\psi_j^*\rangle$ and $\langle\psi_i,\psi_j^*\rangle = 0 = \langle\psi_i^*,\psi_j\rangle$.

coefficients it can be shown that the map between the two vacua at $\mathscr{I}^\pm$ is given by (see for instance section 7.3 in the black holes lecture notes gr-qc/9707012 or section 8.2 in the textbook "Introduction to Quantum Effects in Gravity" by Mukhanov and Winitzki)

$$B_{\omega,\tilde{\omega}} = e^{-\pi\omega/\kappa}\, A_{\omega,\tilde{\omega}} \tag{23}$$

where κ is surface gravity of the black hole horizon.

Inserting the result (23) into the left Bogoliubov relation (16) yields a chain of equalities,

$$\delta_{\omega,\tilde{\omega}} = \sum_\lambda \big(A_{\omega,\lambda}A^*_{\tilde{\omega},\lambda} - B_{\omega,\lambda}B^*_{\tilde{\omega},\lambda}\big) = \Big(e^{\pi(\omega+\tilde{\omega})/\kappa} - 1\Big) \sum_\lambda B_{\omega,\lambda}B^*_{\tilde{\omega},\lambda} \tag{24}$$

that establishes

$$\big(BB^\dagger\big)_{\omega,\omega} = \frac{1}{e^{2\pi\omega/\kappa} - 1}\,. \tag{25}$$

Since the B-coefficients are non-zero we have thus **particle creation by black holes**.

To check what spectrum we obtain we calculate the vacuum expectation value of the number operator in the vacuum near $\mathscr{I}^+$, using (21) and (25).

$$\langle 0_{\mathscr{I}^+}|N_\omega|0_{\mathscr{I}^+}\rangle = \frac{1}{e^{2\pi\omega/\kappa} - 1} \tag{26}$$

This is nothing but the **Planck distribution for a black body at the Hawking temperature** (9)**.**

There are alternative semi-classical derivations. A nice one is given in Unruh's paper, *Phys. Rev.* **D14** (1976) 870. Another efficient method is to derive the existence of a vacuum expectation value of the stress tensor from anomalies. Let me sketch here one such derivation that works for black holes that are effectively two-dimensional (including Schwarzschild). Imposing conformal gauge

$$\mathrm{d}s^2 = e^{2\Omega}\, 2\,\mathrm{d}x^+\,\mathrm{d}x^- \tag{27}$$

covariant conservation of the vev of the stress tensor, $\nabla_\mu\langle T^{\mu\nu}\rangle = 0$, viz.

$$\partial_+\langle T_{--}\rangle + \partial_-\langle T_{+-}\rangle - 2(\partial_-\Omega)\langle T_{+-}\rangle = 0 \tag{28}$$

allows to determine the flux component $\langle T_{--}\rangle$ from the trace component $\langle T_{+-}\rangle$ up to an integration constant, since on static backgrounds $\partial_+ = -\partial_- = \partial_z/\sqrt{2} = \xi(r)\partial_r/\sqrt{2}$ (the same remarks and calculations apply to $\langle T_{++}\rangle$, which for brevity we do not display). A straightforward (but for these lecture notes too lengthy) 1-loop calculation yields the trace anomaly $\langle T^\mu_\mu\rangle \propto R$, which leads to $\langle T_{+-}\rangle = \partial_z^2\Omega/(24\pi)$ and establishes

$$\langle T_{--}\rangle = \frac{1}{24\pi}\left(\partial_z^2\Omega - (\partial_z\Omega)^2\right) + t_- \tag{29}$$

where the integration constant t_- is fixed by the regularity requirement $\langle T_{--}\rangle = 0$ at the horizon (otherwise infinite blueshift factors would render the flux component singular at the horizon in global coordinates). By virtue of $\Omega = \frac{1}{2}\ln\xi$, with ξ being the Killing norm, this constant is fixed as (for the second equality recall exercise 8.2 of Black Holes I)

$$t_- = \frac{(\partial_r\xi)^2\big|_{r=r_{\textrm{horizon}}}}{96\pi} = \frac{\kappa^2}{24\pi} = \frac{\pi}{6}\left(\frac{\kappa}{2\pi}\right)^2 = \frac{\pi}{6}\,T^2 \tag{30}$$

where T is the Hawking temperature (9). The result (30) gives the asymptotic energy flux and is compatible with the two-dimensional version of the Stefan–Boltzmann law. See section 6 in hep-th/0204253 for more on this derivation and on details how to calculate the trace anomaly using heat kernel methods.