8 Black hole thermodynamics

The insight that black holes have a (Hawking) temperature and a (Bekenstein–Hawking) entropy has profoundly influenced our understanding of black holes and our path on the road towards quantum gravity. Despite of their classical simplicity, captured by “no hair” theorems, quantum mechanically black holes are not only complicated, but arguably the most complex entities that could possibly exist in our (or any other) Universe. Sometimes the analogy is made that understanding the thermodynamics of black holes quantum mechanically could play the same role for the development of quantum gravity as the quantum mechanical understanding of the Hydrogen atom in the development of quantum mechanics. Regardless of whether this turns out to be true, black hole thermodynamics certainly is a cornerstone in reasonable attempts to quantize gravity and has found applications in AdS/CFT and black hole analogs.

The reason why there are no astrophysical applications in the current phase of our Universe is the smallness of the Hawking temperature for black holes whose mass is larger than the mass of our Sun (which applies to all astrophysical black holes detected so far and must be true if the black hole results from gravitational collapse of a star, see the beginning of Black Holes I).

In this section we work out classical aspects of black hole thermodynamics, starting with the four laws.

See 1402.5127 and Refs. therein for more on black hole thermodynamics.

8.1 Four laws of black hole mechanics and thermodynamics

In Black Holes I we derived a version of the zeroth law of black hole mechanics (surface gravity \( \kappa \) is constant for stationary black holes) and in Black Holes II we mentioned the proof idea of the second law of black hole mechanics. The third law states that it is impossible to reach a black hole state of vanishing surface gravity from an initial black hole with non-vanishing surface gravity in finite time (see one of the exercises). We focus now on the missing item, the first law of black hole mechanics.

As a preparation we consider Smarr’s formula

\[
M = \frac{\kappa A}{4\pi} + 2\Omega J
\]

for Kerr black holes with mass \( M \), angular momentum \( J \), event horizon area \( A \), surface gravity \( \kappa \) and angular velocity of the horizon \( \Omega \). Smarr’s formula can be derived using the Komar integrals we introduced in Black Holes I for the Killing vector \( \partial_t + \Omega \partial_\phi \), but since we know already all the results for Kerr black holes we can easily verify (1) simply by expressing all quantities in terms of outer and inner horizon radii. Recalling

\[
M = \frac{r_+ + r_-}{2}, \quad J(= aM) = \frac{r_+ + r_-}{2} \sqrt{r_+ r_-}, \quad A = 4\pi \left( r_+^2 + r_+ r_- \right)
\]

\[
\kappa = \frac{r_+ - r_-}{2(r_+^2 + r_+ r_-)}, \quad \Omega = \frac{\sqrt{r_+ r_-}}{r_+^2 + r_+ r_-}
\]

allows to verify that (1) indeed holds for all values of \( r_\pm \).

We state now a simplified version of the first law. Take a stationary (Kerr) black hole of mass \( M \) and angular momentum \( J \) and perturb it infinitesimally by changing to mass \( M + \delta M \) and angular momentum \( J + \delta J \). Then the change of the area \( \delta A \) is related linearly to \( \delta M \) and \( \delta J \) through the first law as follows,

\[
\delta M = \frac{\kappa}{8\pi} \delta A + \Omega \delta J
\]
where $\kappa$ is surface gravity and $\Omega$ the angular velocity of the horizon. The proof of the first law can be found in the paper by Bardeen, Carter and Hawking.\footnote{This paper is not only nice, but also remarkable since it contains the statement “In fact the effective temperature of a black hole is absolute zero. One way of seeing this is to note that a black hole cannot be in equilibrium with black body radiation at any non-zero temperature, because no radiation could be emitted from the hole whereas some radiation would always cross the horizon into the black hole.” that was famously falsified by its last author about a year later.}

Instead of an actual proof we present here a slick derivation that is due to Gibbons. Black hole uniqueness implies that $M$, $J$ and $A$ cannot be independent from each other since (Kerr) black holes are uniquely specified by providing two of these numbers. Thus, either of them must be a function of the two others. For instance, $M = M(J, A)$. Now, $A$ and $J$ have both dimensions of mass squared (we are in four spacetime dimensions right now). This implies that the function $M(J, A)$ is homogeneous of degree $\frac{1}{2}$. Euler’s theorem for homogeneous functions establishes

$$J \frac{\partial M}{\partial J} + A \frac{\partial M}{\partial A} = \frac{1}{2} M = \frac{\kappa}{8\pi} A + \Omega J$$

(4)

where the last equality follows from Smarr’s formula (1). We can rewrite (4) suggestively

$$J \left( \frac{\partial M}{\partial J} - \Omega \right) + A \left( \frac{\partial M}{\partial A} - \frac{\kappa}{8\pi} \right) = 0$$

(5)

and then argue that both terms in (5) have to vanish separately since the coefficients $J$ and $A$ are arbitrary and independent from each other. If you buy this argument then you obtain the desired result

$$\frac{\partial M}{\partial J} = \Omega \quad \frac{\partial M}{\partial A} = \frac{\kappa}{8\pi}$$

(6)

which establishes the first law (3).

More general black holes may also depend on the electric charge and be immersed in something other than Minkowski space, e.g. in (A)dS space. In all these cases there is a first law of the form

$$\delta M = \frac{\kappa}{8\pi} \delta A + \text{work terms}.$$ \hspace{1cm} (7)

We summarize now the four laws of black hole mechanics and contrast them with the four laws of thermodynamics.

<table>
<thead>
<tr>
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<th>black hole mechanics</th>
<th>thermodynamics</th>
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<tbody>
<tr>
<td>0(^{th})</td>
<td>$\kappa = \text{const.}$</td>
<td>$T = \text{const.}$</td>
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<tr>
<td>1(^{st})</td>
<td>$\delta M = \frac{\kappa}{8\pi} \delta A + \text{work terms}$</td>
<td>$\delta E = T \delta S + \text{work terms}$</td>
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<tr>
<td>2(^{nd})</td>
<td>$\delta A \geq 0$</td>
<td>$\delta S \geq 0$</td>
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<tr>
<td>3(^{rd})</td>
<td>$\kappa \rightarrow 0 \text{ impossible}$</td>
<td>$T \rightarrow 0 \text{ impossible}$</td>
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Comparing left and right columns it is tempting to identify surface gravity with temperature, $\kappa \sim T$, area with entropy, $A \sim S$ and mass with energy, $M \sim E$. Actually, we know that at least the last identification is correct, thanks to Einstein’s most famous formula $E = M$ (in units of $c = 1$). Moreover, note that there are non-trivial consistency checks of this identification — for example, $\kappa$ plays the role of $T$ not only in the 0\(^{th}\) law, but also in the 1\(^{st}\) and 3\(^{rd}\) law. Should we therefore take the analogy displayed in the table above seriously? The naive answer is yes, the more sophisticated answer is no (see the footnote on this page for the reason) and the correct answer is again yes. However, to show this we need to take into account quantum fluctuations on black hole backgrounds in order to derive the Hawking effect, the Hawking–Unruh temperature and the Bekenstein–Hawking entropy.
8.2 Phenomenological aspects of black hole thermodynamics

Before we delve into semi-classical aspects associated with the Hawking effect we address phenomenological aspects of black hole thermodynamics. Let us start with the Schwarzschild black hole. According to our previous discussion we have

\[ T \sim \frac{1}{M} \quad S \sim M^2 \quad \Rightarrow \quad S \sim \frac{1}{T^2} \]  

(8)

where the similarity signs remind us that we do not know the factors of order unity in these identifications (all we know is \( \kappa A = 8\pi TS \)). Interesting observations:

1. For stellar mass black holes the temperature is tiny, \( T \sim 10^{-38} \approx 10^{-6} \) Kelvin \( \ll T_{\text{CMB}} \approx 3 \) Kelvin. (Once all factors are considered the result for a stellar mass black hole is \( T \approx 61.7 \) nanoKelvin.) Thus, we do not expect to ever detect Hawking radiation from stellar mass black holes (nor from heavier ones).

2. For a stellar mass black hole the entropy is ridiculously large, \( S \sim 10^{76} \), which means that we have a googolplex-like number of microstates, \( N \sim e^{10^{76}} \).

3. The Bekenstein–Hawking entropy is not extensive in the usual way, i.e., it does not scale like the volume of the black hole but rather like its area. This observation is the seed of the holographic principle, which states that quantum gravity in, say, four spacetime dimensions is equivalent to some quantum field theory in three spacetime dimensions (where then the area is reinterpreted as volume of the lower-dimensional theory).

4. The Schwarzschild black hole has negative specific heat.

\[ C = \frac{dM}{dT} \sim -\frac{1}{T^2} < 0 \]  

(9)

This statement just rephrases the fact that the more a Schwarzschild black hole radiates (and hence the more it reduces its mass) the warmer it gets. Thus, by itself the Schwarzschild black hole is thermodynamically unstable, but we should not worry too much about this given how tiny the specific heat is. It is possible to stabilize the Schwarzschild black hole by putting it into a box (either literally or by providing AdS asymptotics, see below).

Charged (Reissner–Nordström) or rotating (Kerr or Kerr–Newman) black holes have additional interesting features. There are now work terms present associated with changes of the charge or angular momentum. Moreover, we can have extremal solutions where temperature vanishes, but which are macroscopically large and thus have a huge entropy. For example,

\[ S_{\text{extremal Kerr}} \sim A = 4\pi (r_+^2 + r_-^2) = 8\pi r_+^2 = 8\pi M^2 \gg 1. \]  

(10)

No analog condensed matter system is known which at zero temperature has such a large degeneracy of states.

Finally, let us briefly consider black holes in AdS; for simplicity consider Schwarzschild-AdS, whose metric is given by (\( \ell \) is the AdS\(_4\) radius)

\[ ds^2 = -\left( \frac{r^2}{\ell^2} + 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2} + 1 - \frac{2M}{r}} + r^2 d\Omega_5^2. \]  

(11)

Calculating the specific heat in the limit of small masses recovers the negative sign of (9), \( C \sim -M^2 + \mathcal{O}(M^4/\ell^2) \). Interestingly, in the limit of large masses specific heat is positive, \( C \sim \ell^{4/3} M^{2/3} + \mathcal{O}(M^{8/3}/M^{2/3}) \). This suggests that there could be a phase transition at some finite value of the mass, \( M/\ell \sim \mathcal{O}(1) \), which indeed exists and is known as Hawking–Page phase transition.