4 Linearized gravity

In many instances (not just in gravity but also in quantum field theory) one is interested in linearizing perturbations around a fixed background, which considerably simplifies the classical and quantum analysis. While this approach is only justified if the linearized perturbation is small enough, there are numerous applications where this assumption holds. Examples include gravitational waves, holographic applications and perturbative quantization of gravity. In this section we develop the basic tools to address all these issues.

4.1 Linearization of geometry around fixed background

Assume that the metric can be meaningfully split into background $\bar{g}_{\mu\nu}$ and fluctuations $h_{\mu\nu}$. You can think of $\bar{g}$ as some classical background (e.g. Minkowski space, AdS, dS, FLRW or some black hole background) and of h either as a classical perturbation (e.g. a gravitational wave on your background) or as a variation of the metric (e.g. when checking the variational principle or in holographic contexts) or as a quantum fluctuation (e.g. when semi-classically quantizing gravity).

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{1}$$

For calculations we generally need various geometric quantities, like the inverse metric, the Christoffel symbols, the Riemann tensor etc., so we consider them now to linear order in h. Note that $h_{\mu\nu} = h_{\nu\mu}$ is a symmetric tensor.

Let us start with the inverse metric. The identity $g^{\mu\nu}g_{\nu\lambda} = \delta^\mu_\lambda$ yields

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2)\,. \tag{2}$$

In all linearized expressions we raise and lower indices with the background metric $\bar{g}$, so that e.g. $h^{\mu\nu} = \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}h_{\alpha\beta}$. All quantities with bar on top have their usual meaning and are constructed from the background metric $\bar{g}$, e.g. $\bar{\Gamma}^\alpha{}_{\beta\gamma} = \frac{1}{2}\,\bar{g}^{\alpha\mu}\left(\bar{g}_{\beta\mu,\gamma} + \bar{g}_{\gamma\mu,\beta} - \bar{g}_{\beta\gamma,\mu}\right)$. We denote the difference between full and background expression with δ, for example $\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu} = h_{\mu\nu}$ and $\delta g^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu} = -h^{\mu\nu}$.

The determinant of the metric expands as explained in Black Holes I. (We suppress from now on $\mathcal{O}(h^2)$ as it is understood that all equations below hold only at linearized level.)

$$\sqrt{-g} = \sqrt{-\bar{g}}\left(1 + \frac{1}{2}\,\bar{g}^{\mu\nu}h_{\mu\nu}\right) \tag{3}$$

The Christoffel symbols expand as follows

$$\delta\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} - \bar{\Gamma}^\alpha{}_{\beta\gamma} = \frac{1}{2}\,\bar{g}^{\alpha\mu}\left(\bar{\nabla}_\beta h_{\gamma\mu} + \bar{\nabla}_\gamma h_{\beta\mu} - \bar{\nabla}_\mu h_{\beta\gamma}\right). \tag{4}$$

The result (4) implies that the **variation of the Christoffels, $\delta\Gamma$, is a tensor**.

The linearized Riemann tensor can be expressed concisely in terms of (4).

$$\boxed{\delta R^\alpha{}_{\beta\mu\nu} = \bar{\nabla}_\mu\,\delta\Gamma^\alpha{}_{\beta\nu} - \bar{\nabla}_\nu\,\delta\Gamma^\alpha{}_{\beta\mu}} \tag{5}$$

While the results above are all we need for now, it is useful to provide more explicit results for the linearized Ricci-tensor

$$\delta R_{\mu\nu} = \bar{\nabla}_\alpha\,\delta\Gamma^\alpha{}_{\mu\nu} - \bar{\nabla}_\nu\,\delta\Gamma^\alpha{}_{\mu\alpha} = \frac{1}{2}\left(\bar{\nabla}^\alpha\bar{\nabla}_\mu h_{\alpha\nu} + \bar{\nabla}^\alpha\bar{\nabla}_\nu h_{\alpha\mu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h^\alpha_\alpha - \bar{\nabla}^2 h_{\mu\nu}\right) \tag{6}$$

and the linearized Ricci-scalar

$$\delta R = -\bar{R}^{\mu\nu}h_{\mu\nu} + \bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h^\mu_\mu\,. \tag{7}$$

4.2 Linearization of Einstein equations

Consider the vacuum Einstein equations $R_{\mu\nu} = 0$ and assume some solution thereof for the background metric, $\bar{g}_{\mu\nu}$ such that $\bar{R}_{\mu\nu} = 0$ (e.g. $\bar{g}$ could be Minkowski space or the Kerr solution). Classical perturbations around that background then have to obey the linearized Einstein equations $\delta R_{\mu\nu} = 0$, viz.

$$\bar{\nabla}^\alpha \bar{\nabla}_\mu h_{\alpha\nu} + \bar{\nabla}^\alpha \bar{\nabla}_\nu h_{\alpha\mu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h^\alpha_\alpha - \bar{\nabla}^2 h_{\mu\nu} = 0\,. \tag{8}$$

Before attempting to solve these equations it is useful to decompose the perturbations h as follows.

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + \bar{\nabla}_{(\mu}\xi_{\nu)} + \frac{1}{D}\,\bar{g}_{\mu\nu}\,h \tag{9}$$

The first contribution on the right hand side of (9) is called "transverse-traceless part" (TT-part) since it obeys the conditions

$$\bar{\nabla}^\mu h^{TT}_{\mu\nu} = 0 = h_\mu^{\mu\,TT}\,. \tag{10}$$

The second contribution on the right hand side of (9) is called "gauge part" since it can be compensated by an infinitesimal diffeomorphism of the background metric, $\mathcal{L}_\xi \bar{g}_{\mu\nu} = \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$. The last contribution on the right hand side of (9) is called "trace part", since up to a gauge term the trace of $h_{\mu\nu}$ is given by h. Alternatively, one can call the three contributions (in this order) tensor, vector and scalar part.

In $D \geq 3$ spacetime dimensions the tensor $h_{\mu\nu}$ has $D(D+1)/2$ algebraically independent components, with D of them residing in the gauge part and 1 of them in the trace part. This means at this stage the TT-part has $(D+1)(D-2)/2$ algebraically independent components, which corresponds to the correct number of massive spin-2 polarizations. However, in Einstein gravity gravitons are massless which reduces the number of polarizations. As we shall see below **there are $D(D-3)/2$ gravity wave polarizations in D-dimensional Einstein gravity.**

For simplicity we assume from now on that the background metric is flat so that $\bar{R}^\alpha{}_{\beta\gamma\delta} = 0$. We evaluate for this case the linearized Einstein equations (8) separately for the TT-part[1]

$$\text{on flat background:}\quad \bar{\nabla}^2 h^{TT}_{\mu\nu} = 0 \tag{11}$$

and the trace part $(\bar{g}_{\mu\nu}\bar{\nabla}^2 + (D-2)\bar{\nabla}_\mu\bar{\nabla}_\nu)h = 0$. The gauge part trivially solves the linearized Einstein equations (8).

Thus, on a flat background the TT-part obeys a wave equation (11), essentially of the same type as a vacuum Maxwell-field in Lorenz-gauge. We show now that the same wave equation can be obtained by suitable gauge fixing of the original $h_{\mu\nu}$, namely by imposing harmonic gauge, a.k.a. de-Donder gauge

$$\bar{\nabla}_\mu h^\mu_\nu = \frac{1}{2}\,\partial_\nu h^\mu_\mu\,. \tag{12}$$

The gauge choice (12) fixes D of the $D(D+1)/2$ components of $h_{\mu\nu}$, but we still have residual gauge freedom, i.e., gauge transformations

$$h_{\mu\nu} \to \tilde{h}_{\mu\nu} = h_{\mu\nu} + \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu \quad\text{such that}\quad \bar{\nabla}_\mu \tilde{h}^\mu_\nu = \frac{1}{2}\,\partial_\nu \tilde{h}^\mu_\mu \tag{13}$$

that preserve de-Donder gauge. The last equality in (13) establishes $\bar{\nabla}^2\xi_\mu = 0$, so that we have D independent residual gauge transformations. In conclusion, the number of physical degrees of freedom contained in linearized perturbations $h_{\mu\nu}$ in Einstein gravity is given by $D(D+1)/2 - 2D = D(D-3)/2$. Inserting de-Donder gauge (12) into the linearized Einstein equations (8) yields $\bar{\nabla}^2 h_{\mu\nu} = 0$, as promised.

[1] The reason why this makes sense is because TT-, gauge- and trace-part decouple in the quadratic action (16) below. Hence, also the linearized field equations decouple.

4.3 Linearization of Hilbert action

We can use the linearization not only at the level of field equations but also at the level of the action.

As a first task we fill in a gap that was left open in Black Holes I when deriving the Einstein equations from varying the Hilbert action. We drop here all bars on top of the metric and denote the fluctuation by δg instead of h. Using the formulas for the variation of the determinant (3) and the Ricci scalar (7) yields

$$\delta I_{\text{EH}} = \frac{1}{16\pi G}\,\delta\int \mathrm{d}^D x\sqrt{-g}\,R = \frac{1}{16\pi G}\int \mathrm{d}^D x\sqrt{-g}\,\Big(\big(\tfrac{1}{2}\,g^{\mu\nu}\,R - R^{\mu\nu}\big)\,\delta g_{\mu\nu} + \nabla^\mu\big(\nabla^\nu\delta g_{\mu\nu} - g^{\alpha\beta}\nabla_\mu\delta g_{\alpha\beta}\big)\Big)\,. \tag{14}$$

Setting to zero the terms in the first line for arbitrary variations yields the vacuum Einstein equations. The terms in the second line are total derivative terms and vanish upon introducing a suitable boundary action and suitable boundary conditions on the metric (we shall learn more about this later in these lectures).

As second task we vary the action (14) again to obtain an expression quadratic in the fluctuations (again dropping total derivative terms). Since $\delta^2 g_{\mu\nu} = 0$ and the Einstein equations hold for the background we only need to vary the Einstein tensor.

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \tfrac{1}{2}\,\bar{R}\delta g_{\mu\nu} - \tfrac{1}{2}\,\bar{g}_{\mu\nu}\,\delta R = \delta R_{\mu\nu} - \tfrac{1}{2}\,\bar{g}_{\mu\nu}\,\delta R \tag{15}$$

Plugging (15) together with (6) and (7) into the second variation of the action (using again h instead of δg) establishes the quadratic action (up to boundary terms)

$$16\pi G\, I^{(2)}_{\text{EH}} = \int \mathrm{d}^D x\sqrt{-\bar{g}}\,h^{\mu\nu}\,\delta G_{\mu\nu} = \int \mathrm{d}^D x\sqrt{-\bar{g}}\,\tfrac{1}{2}\,h^{\mu\nu}\,\big(\bar{\Box}_{\mu\nu}{}^{\alpha\beta}\,h_{\alpha\beta}\big) \tag{16}$$

with the wave operator

$$\bar{\Box}_{\mu\nu}{}^{\alpha\beta} = \delta^\beta_\nu\bar{\nabla}^\alpha\bar{\nabla}_\mu + \delta^\beta_\mu\bar{\nabla}^\alpha\bar{\nabla}_\nu - \bar{g}^{\alpha\beta}\,\bar{\nabla}_\mu\bar{\nabla}_\nu - \delta^\alpha_\mu\delta^\beta_\nu\bar{\nabla}^2 - \bar{g}_{\mu\nu}\,\bar{\nabla}^\alpha\bar{\nabla}^\beta + \bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\,\bar{\nabla}^2\,. \tag{17}$$

The quadratic action (16) has a number of uses for semi-classical gravity and holography. The field equations for h associated with the action (16), $\bar{\Box}_{\mu\nu}{}^{\alpha\beta}\,h_{\alpha\beta} = 0$, are equivalent to the linearized Einstein equations (8). Thus, the action (16) is a perturbative action for the gravitational wave (plus gauge) degrees of freedom.

4.4 Backreactions and recovering Einstein gravity

In this subsection we work schematically, omitting factors and indices. In the presence of matter sources T the quadratic action reads $I^{(2)} \sim \int(\frac{1}{G}\,h\partial^2 h + hT)$. However, in contrast to electrodynamics where the photon is not charged, the graviton is charged under its own gauge group, i.e., gravitons have energy and thus interact with themselves. One can take this effect into account perturbatively by calculating the energy-momentum tensor associated with the quadratic fluctuations, which schematically is of the form $T^{(2)} \sim \frac{1}{G}\,\partial h\partial h$. Thus, taking into account backreactions we are led to a cubic action $I^{(3)} \sim \int(\frac{1}{G}h\partial^2 h + \frac{1}{G}\,h\partial h\partial h + hT)$. However, the cubic term also contributes to the stress tensor, $T^{(3)} \sim \frac{1}{G}\,h\partial h\partial h$ and so forth. Continuing this perturbative expansion yields an action

$$I^{(\infty)} \sim \int \big[\tfrac{1}{G}\,\big(h\partial^2 h + h\partial h\partial h + h^2\partial h\partial h + h^3\partial h\partial h + \dots\big) + hT\big]\,. \tag{18}$$

It was shown by Boulware and Deser that the whole sum can be rewritten as $\frac{1}{G}\,\sqrt{-g}\,R$, so that even if one had never heard of Riemannian geometry in principle one could derive the Hilbert action of Einstein gravity by starting with a massless spin-2 action (16), adding a source and taking into account consistently backreactions.

5 Gravitational waves

5.1 Gravitational waves in vacuum

Let us stick to $D = 4$ and solve the gravitational wave equation on a Minkowski background together with de-Donder gauge,

$$\partial^2 h_{\mu\nu} = 0 = \partial_\mu h^\mu_\nu - \tfrac{1}{2}\,\partial_\nu h^\mu_\mu\,. \tag{19}$$

Linearity of the wave equation allows us to use the superposition principle and build the general solution in terms of plane waves

$$h_{\mu\nu} = \epsilon_{\mu\nu}(k)\, e^{ik_\mu x^\mu} \qquad\qquad k^2 = 0 \qquad\qquad k_\mu \epsilon^\mu_\nu = \tfrac{1}{2}\, k_\nu \epsilon^\mu_\mu\,. \tag{20}$$

The first equality contains the symmetric polarization tensor $\epsilon_{\mu\nu}$ that has to obey the third equality to be compatible with de-Donder gauge. The second equality ensures that the wave equation holds. The general solution is then some arbitrary superposition of plane waves (20), exactly as for photons in electrodynamics.

The four residual gauge transformations are now used to set to zero the components $\epsilon_{0i} = 0$ and the trace $\epsilon^\mu_\mu = 0$. Thus, the de-Donder condition (20) simplifies to transversality, $k^\mu \epsilon_{\mu\nu} = 0$. [With these choices the polarization tensor is transverse and traceless, so that only h^{TT} in (9) contributes.] For concreteness assume now that the gravitational wave propagates in z-direction, $k^\mu = \omega\,(1, 0, 0, 1)^\mu$. Then transversality implies $\epsilon_{00} = \epsilon_{0x} = \epsilon_{0y} = \epsilon_{0z} = \epsilon_{xz} = \epsilon_{yz} = \epsilon_{zz} = 0$. Together with symmetry, $\epsilon_{\mu\nu} = \epsilon_{\nu\mu}$, and traceleceness, $\epsilon^\mu_\mu = 0$, the polarization tensor

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_+ & \epsilon_\times & 0 \\ 0 & \epsilon_\times & -\epsilon_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu} =: \epsilon^+_{\mu\nu} + \epsilon^\times_{\mu\nu} \tag{21}$$

is characterized by two real numbers, corresponding to the two polarizations of gravitational waves or, equivalently, to the two helicity states of massless spin-2 particles. They are called "plus-polarization" (ϵ_+) and "cross-polarization" ($\epsilon_\times$).

5.2 Gravitational waves acting on test particles

With a single test-particle it is impossible to detect a gravitational wave, so let us assume there are two massive test-particles, one at the origin (A) and the other (B) at some finite distance L_0 along the x-axis. Let us further assume there is a planar gravitational wave propagating along the z-direction with +-polarization, $h_{\mu\nu} = \epsilon^+_{\mu\nu} f(t - z)$ with $\epsilon_+ = 1$. The perturbed metric then reads

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \big(1 + f(t - z)\big)\,\mathrm{d}x^2 + \big(1 - f(t - z)\big)\,\mathrm{d}y^2 + \mathrm{d}z^2 \qquad f \ll 1\,. \tag{22}$$

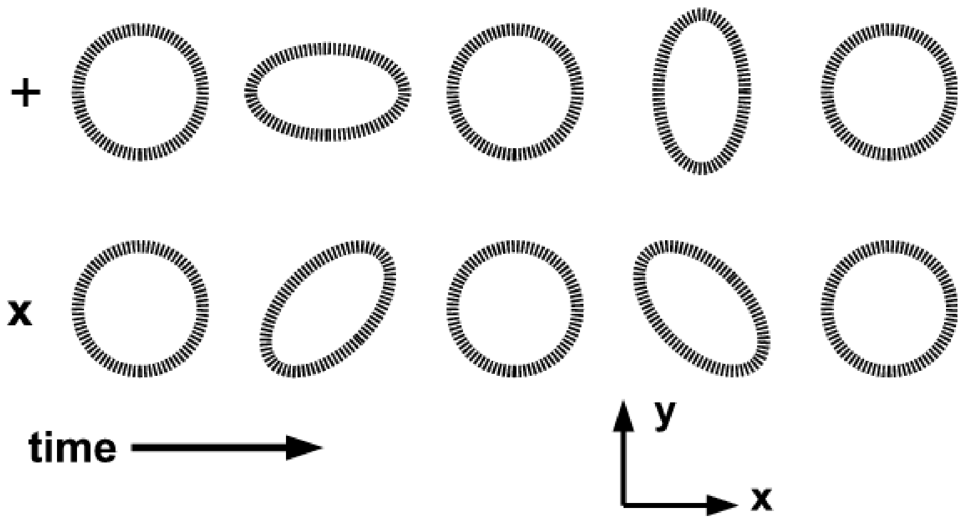
Assuming both test-particles are at rest originally, $u^\mu_A = u^\mu_B = (1, 0, 0, 0)$, we can solve the geodesic equation to linearized order.

$$\mathrm{d}u^\mu / \mathrm{d}\tau = -\delta\Gamma^\mu{}_{00} = 0 \tag{23}$$

The last equality is checked easily by explicitly calculating the relevant Christoffel symbols for the metric (22). Since the right hand side in (23) vanishes the test-particles remain at rest and the coordinate distance between A and B does not change. However, the proper distance between them changes (we keep $y = z = 0$).

$$L(t) = \int_0^{L_0} \mathrm{d}x\,\sqrt{1 + f(t)} \qquad \Rightarrow \qquad \frac{L(t) - L_0}{L_0} \approx \frac{1}{2}\, f(t) \tag{24}$$

For periodic functions f the proper distance thus oscillates periodically around its mean value L_0. This is an effect that in principle can be measured, e.g. with LIGO.



Effects of plus and cross polarized gravitational waves on ring of test-particles

5.3 Gravitational wave emission

Like light-waves, gravitational waves need a source. In the former case the source consists of accelerated charges, producing dipole (and higher multipole) radiation, in the latter case the source consists of energy, producing quadrupole (and higher multipole) radiation. The first step is to generalize the wave equation (19) (defining $\tilde{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,h^{\alpha}_{\alpha}$ so that de-Donder gauge reads $\partial^{\mu}\tilde{h}_{\mu\nu} = 0$) to include an energy-momentum tensor as source

$$\partial^2\tilde{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu} \tag{25}$$

which for consistency has to obey the conservation equation $\partial^{\mu}T_{\mu\nu} = 0$.

Up to the decoration with an additional index this is precisely the same situation as in electrodynamics, where the inhomogeneous Maxwell-equations in Lorenz-gauge read $\partial^2 A_{\mu} = -4\pi\, j_{\mu}$ and the source has to obey the conservation equation $\partial^{\mu} j_{\mu} = 0$. Using the retarded Green function yields

$$\tilde{h}_{\mu\nu}(t,\,\vec{x}) = 4G\int \mathrm{d}^3x'\,\frac{T_{\mu\nu}(t-|\vec{x}-\vec{x}'|,\,\vec{x}')}{|\vec{x}-\vec{x}'|}\,. \tag{26}$$

Thus, we can basically apply nearly everything we know from electrodynamics to gravitational waves. We shall not do this here in great detail, but consider merely one example, the multipole expansion. Taylor-expanding around $\vec{x}' = 0$ the factor $|\vec{x}-\vec{x}'| = r(1-\vec{x}\cdot\vec{x}'/r^2+\dots)$ in (26) yields

$$\frac{\tilde{h}_{\mu\nu}(t,\,\vec{x})}{4G} = \frac{1}{r}\int T_{\mu\nu} + \frac{x^i}{r^3}\int x'_i T_{\mu\nu} + \frac{3x^i x^j - r^2\delta^{ij}}{2r^5}\int x'_i x'_j T_{\mu\nu} + \dots \tag{27}$$

The quantities $\int T^{00} = \int \mathrm{d}^3x'\, T^{00}(t-|\vec{x}-\vec{x}'|,\vec{x}') = M$ and $\int T^{0i} = \int \mathrm{d}^3x'\, T^{0i}(t-|\vec{x}-\vec{x}'|,\vec{x}') = P^i$ are mass and momentum of the source. A few lines of calculation establish a formula for $\tilde{h}^{ij}$ in terms of the second time-derivative of the quadrupole moment $Q^{ij}(t) := \int \mathrm{d}^3x' x'^i x'^j T^{00}(t,\,\vec{x}')$ of the source.

$$\tilde{h}^{ij}(t,\,\vec{x}) = \frac{2G}{r}\,\frac{\mathrm{d}^2Q^{ij}(t)}{\mathrm{d}t^2}\Big|_{t\to t-|\vec{x}-\vec{x}'|} \tag{28}$$

In the far-field approximation (28) describes the dominant part of gravitational radiation.

6 Quantum field theory aspects of spin-2 particles

There are undeniable analogies between Maxwell's theory (a theory of massless spin-1 fields), with the linearized gauge symmetry

$$A_\mu \to A_\mu + \partial_\mu \xi \tag{29}$$

and linearized Einstein gravity on Minkowski background (a theory of massless spin-2 fields), with the linearized gauge symmetry

$$\boxed{h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}\,.} \tag{30}$$

(This analogy extends to spins higher than 2.) In the remainder of this section we work exclusively in four spacetime dimensions for sake of specificity.

6.1 Gravitoelectromagnetism

As we have shown in section 4.2 in a suitable gauge $h_{\mu\nu}$ obeys the same wave equation as A_μ. In fact, given some observer worldline u^μ one can do a split analogous to electromagnetism into electric part and magnetic part of the Weyl tensor (the Ricci tensor vanishes for vacuum solutions), which in $D = 4$ reads

$$E_{\mu\nu} = C_{\mu\alpha\nu\beta}\, u^\alpha u^\beta \qquad\qquad B_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\alpha}{}^{\lambda\gamma}\, C_{\nu\beta\lambda\gamma}\, u^\alpha u^\beta\,. \tag{31}$$

If you want to read more on this formulation see for instance in gr-qc/9704059.

6.2 Massive spin-2 QFT

We can gain some insights from looking at the quantum field theory of spin-1 particles (massless or massive QED) and extrapolating results to massless or massive spin-2 particles. (If you are unfamiliar with QED just skip the remainder of this section.) A particular goal of this subsection is to derive that positive charges repel each other while positive masses attract each other just from the spin of the associated exchange particle (spin-1 for electromagnetism, spin-2 for gravity).

To avoid issues with gauge redundancies consider for the moment the massive case. The effective action for massive spin-1 particles is given by

$$W(j) = -\frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\, j^{\mu\,*}(k)\,\Delta_{\mu\nu}(k)\,j^\nu(k) \tag{32}$$

where j are external currents and $\Delta_{\mu\nu}$ is the propagator,

$$\Delta_{\mu\nu}(k) = \frac{\eta_{\mu\nu} + k_\mu k_\nu/m^2}{k^2 + m^2 - i\epsilon} \tag{33}$$

with the photon mass m and $i\epsilon$ is the prescription to obtain the Feynman propagator. Current conservation $\partial_\mu j^\mu = 0$ implies transversality $k_\mu j^\mu = 0$ so that the second term in the numerator of (33) drops out, yielding

$$W(j) = -\frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\, j^{\mu\,*}(k)\,\frac{1}{k^2 + m^2 - i\epsilon}\,j_\mu(k)\,. \tag{34}$$

Consider now the situation where the sources are stationary charges so that $j^0 \neq 0$ but $j^i = 0$ (assume further that j^0 is real). Then the result above simplifies to

$$W(j^0) = \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,(j^0)^2\,\frac{1}{k^2 + m^2 - i\epsilon}\,. \tag{35}$$

Actually, the only aspect of interest to us is the sign in (35): it is positive, meaning that there is a positive potential energy between charges of the same sign. Thus, equal charges repel each other.

Since we intend to generalize the considerations above to massive spin-2 particles we need to know their propagator. To this end let us rederive the massive photon propagator (33) using transversality of polarization vectors, $k^\mu \epsilon^I_\mu(k) = 0$, where I runs over all possible polarizations (for massive spin-1 particles $I = 1, 2, 3$) and with no loss of generality we choose $k^\mu = m\,(1, 0, 0, 0)$ and $\epsilon^I_\mu = \delta^I_\mu$. On general grounds, the amplitude for creating a state with momentum k and polarization I at the source is proportional to $\epsilon^I_\mu(k)$, and similarly the amplitude for annihilating a state with momentum k and polarization I at the sink is proportional to $\epsilon^I_\nu(k)$. The numerator in (33) (which determines the residue of the poles) should thus be given by the sum $\sum_I \epsilon^I_\mu(k)\epsilon^I_\nu(k)$. Suppose we did not know the result for the residue. Then we can argue that by Lorentz invariance the result must be given by the sum of two terms, one proportional to $g_{\mu\nu}$ and the other proportional to $k_\mu k_\nu$. Transversality fixes the relative coefficient so that the numerator (and hence the residue) must be proportional to

$$D_{\mu\nu} = \eta_{\mu\nu} + k_\mu k_\nu/m^2\,. \tag{36}$$

The overall normalization is determined to be $+1$, e.g. from considering the component $\mu = \nu = 1$. This concludes our derivation of the residue of the pole in the massive spin-1 propagator (33). The location of the pole itself just follows from the wave equation; the $i\epsilon$ prescription is the least obvious aspect, but standard since Feynman's time. If you are unfamiliar with it consult some introductory QFT book, like Peskin & Schroeder.

We do now the same calculation for massive spin-2 particles, where the analog of the effective action (32) reads

$$W(T) = -\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,T^{\mu\nu}(k)\,\Delta_{\mu\nu\alpha\beta}(k)\,T^{\alpha\beta}(k)\,. \tag{37}$$

The source is now the energy-momentum tensor $T^{\mu\nu}$. Source and propagator have twice as many indices as compared to the spin-1 case.

Our first task is to determine the propagator $\Delta_{\mu\nu\alpha\beta}(k)$. We use the spin-2 polarization tensor $\epsilon_{\mu\nu}$, which has to be transverse, traceless and symmetric.

$$k^\mu\epsilon_{\mu\nu} = 0 \qquad \epsilon^\mu{}_\mu = 0 \qquad \epsilon_{\mu\nu} = \epsilon_{\nu\mu} \tag{38}$$

This means that we have 5 independent components in $\epsilon_{\mu\nu}$ corresponding to the 5 spin-2 helicity states. We introduce again a label I to discriminate between these 5 helicity states, $\epsilon^I_{\mu\nu}(k)$ and allow for k-dependence (fixing the normalization e.g. by $\sum_I \epsilon^I_{12}\epsilon^I_{12} = 1$). It is then a straightforward exercise [exploiting the properties (38)] to perform the sum over all helicities

$$\sum_{I=1}^{5}\epsilon^I_{\mu\nu}\epsilon^I_{\alpha\beta} = D_{\mu\alpha}D_{\nu\beta} + D_{\mu\beta}D_{\nu\alpha} - \tfrac{2}{3}\,D_{\mu\nu}D_{\alpha\beta} \tag{39}$$

where $D_{\mu\nu}$ is the same expression as in (36). The overall normalization was fixed again by considering a specific example, e.g. evaluating (39) for $\mu = \alpha = 1$ and $\nu = \beta = 2$. This means that the massive spin-2 (Feynman) propagator is given by

$$\Delta_{\mu\nu\alpha\beta}(k) = \frac{D_{\mu\alpha}D_{\nu\beta} + D_{\mu\beta}D_{\nu\alpha} - \tfrac{2}{3}\,D_{\mu\nu}D_{\alpha\beta}}{k^2 + m^2 - i\epsilon}\,. \tag{40}$$

Our second task is to consider the interaction between two sources of energy. For simplicity assume that only $T^{00} \neq 0$ and all other components of $T^{\mu\nu}$ vanish.

Then inserting the massive spin-2 propagator (40) into the effective action (37) yields [using transversality $k_\mu T^{\mu\nu} = 0$ only the η-term in (36) contributes]

$$W(T^{00}) = -\frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,(T^{00}(k))^2\,\frac{1+1-\frac{2}{3}}{k^2+m^2-i\epsilon}\,. \tag{41}$$

Since all numerator terms in the integrand are positive, the overall sign of the potential energy $W(T^{00})$ is opposite to that of the potential energy $W(j_0)$ in the spin-1 case (35). Thus, **gravity is attractive for positive energy because the exchange particle has spin-2.**

It is remarkable that we were able to conclude the attractiveness of gravity merely from the statement that its exchange particle has spin-2. Of course, there is a gap in the logic above: we have proved this statement so far only for massive gravitons, but Einstein gravity has massless gravitons.

6.3 Massless spin-2 QFT and vDVZ-discontinuity

It may be tempting to conclude that the difference between a massless spin-2 particle and a massive one is negligible if the mass in sufficiently small. Actually this conclusion is correct, but in a highly non-trivial way, which we address here.

Let us consider first the massless spin-2 propagator, which we can read off from the results in section 5.1 [or from (17) together with a gauge-fixing term].

$$\Delta^0_{\mu\nu\alpha\beta}(k) = \frac{\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta}}{k^2+m^2-i\epsilon} + \text{possibly } k_\lambda\text{-terms } (\lambda=\mu,\nu,\alpha,\beta) \tag{42}$$

The main difference to the massive spin-2 propagator (40) is that the factor $-\frac{2}{3}$ in the last term is replaced by -1 here, which causes a discontinuity, as it persists for arbitrarily small non-zero masses. This effect is called **van Dam–Veltman–Zakharov discontinuity**.

Should we care about this discontinuity? Consider the interaction between two particles with stress tensors $T^{\mu\nu}_{1,2}$ exchanging a massive spin-2 particle in the limit of vanishing mass versus them exchanging a massless spin-2 particle:

$$\text{massive } (m\to 0): \qquad T_1^{\mu\nu}\Delta_{\mu\nu\alpha\beta}T_2^{\alpha\beta} = \frac{1}{k^2}\left(2T_1^{\mu\nu}T_{2\,\mu\nu} - \tfrac{2}{3}\,T_1T_2\right) \tag{43}$$

$$\text{massless:} \qquad T_1^{\mu\nu}\Delta^0_{\mu\nu\alpha\beta}T_2^{\alpha\beta} = \frac{1}{k^2}\left(2T_1^{\mu\nu}T_{2\,\mu\nu} - T_1T_2\right) \tag{44}$$

Thus, for the gravitational interaction of massless particles ($T_1 = T_2 = 0$) there is no difference between the exchange of (tiny) massive and massless spin-2 particles, but for massive particles ($T_1 \neq 0 \neq T_2$) there is a difference by a factor of order unity. This factor of order unity should have shown up in the classical tests (light-bending and perihelion shift). So can we conclude from the vDVZ-discontinuity that experimentally the graviton must be exactly massless?

The answer is no. While Einstein gravity predicts massless gravitons, we cannot be sure experimentally whether or not the graviton is exactly massless or has a tiny non-zero mass. The issue why the vDVZ-discontinuity does not contradict this statement was resolved by Vainshtein. His key insight was that massive spin-2 theories with some central object of mass M come with an intrinsic distance scale, given by $r_V = (GM)^{1/5}/m^{4/5}$ also known as "Vainshtein radius" (G is Newton's constant and m the graviton mass). The difference between Einstein gravity and massive spin-2 theories is negligible inside the Vainshtein radius, which can be arbitrarily large if m is tiny. The approximations we made above using massive spin-2 exchange are only valid outside the Vainshtein radius; within the Vainshtein radius the higher order terms in the expansion analogous to (18) are not negligible.