1 Horizons and other definitions

On this sheet several basic definitions regarding the causal structure of spacetime and black holes are summarized. For a more detailed account see Wald's book "General Relativity" (chapters 8-9 and parts of 11-12) or the book by Hawking & Ellis "The large scale structure of space-time".

1.1 Aspects of causal structure of spacetime

Time-like/null/causal curve. A 1-dimensional curve (which may or may not be a geodesic) in some spacetime is called time-like (null) [causal] if the tangent vector is time-like (null) [time-like or null] along the whole curve.

Chronological/causal future of a point p. The chronological future $I^+(p)$ [causal future $J^+(p)$] is the set of all events in spacetime that can be reached from p by a time-like [causal] curve. This definition generalizes to sets of points.

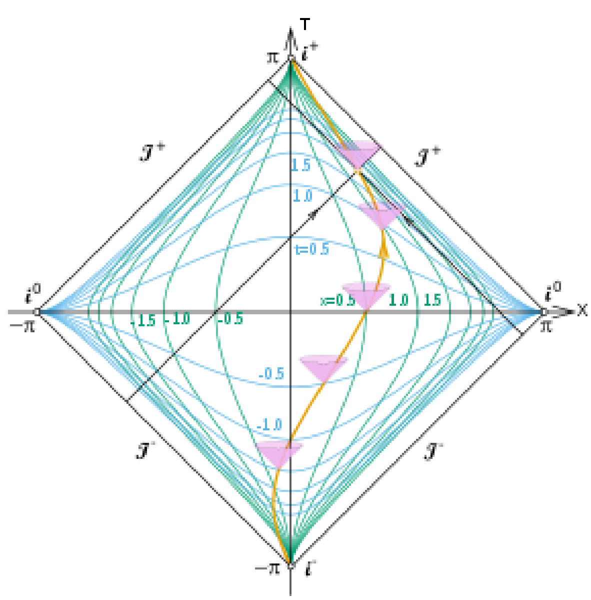
Achronal sets S. A subset S of the spacetime manifold is called achronal if there exists no pair of points $p, q \in S$ such that $q \in I^+(p)$. Equivalently, $I^+(S) \cap S = \{\}$.

Future/past inextendible. A time-like curve is called future (past) inextendible if it has no future (past) endpoint. Analogous definition for causal curves.

Future domain of dependence $D^+(S)$. The future domain of dependence of a closed achronal set S, denoted by $D^+(S)$, is given by the set of all points p in spacetime such that every past inextendible causal curve through p intersects S. Past domain of dependence $D^-(S)$: Exchange "future" $\leftrightarrow$ "past".

Domain of dependence $D(S)$. $D(S) := D^+(S) \cup D^-(S)$.

Asymptotic infinity. Preview of next week on Carter–Penrose diagrams; asymptotic boundaries are denoted by i^+ (future time-like infinity), i^- (past time-like infinity), $\mathscr{I}^+$ (future null infinity), $\mathscr{I}^-$ (past null infinity) and i^0 (spatial infinity.



1.2 Killing, Cauchy, event and apparent horizons

Killing horizon (see last semester). Null hypersurface whose normal is a Killing vector. Useful for stationary black holes, but too restrictive in general.

Cauchy horizon $H(S)$. Let S be a closed achronal set. Its Cauchy horizon $H(S)$ is defined as $H(S) := (\overline{D^+(S)} - I^-[D^+(S)]) \cup (\overline{D^-(S)} - I^+[D^-(S)])$. In plain English, the Cauchy horizon is the boundary of the domain of dependence of S.

Note: A Cauchy horizon is considered as a singularity in the causal structure, since you cannot predict time-evolution beyond a Cauchy horizon.

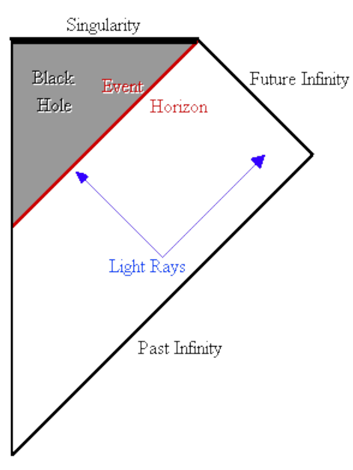
Cauchy surface Σ. A nonempty closed achronal set Σ is a Cauchy surface for some (connected) spacetime manifold iff $H(\Sigma) = \{\}$.

A spacetime M with a Cauchy surface Σ is called "globally hyperbolic".

With the definitions on the first page we are now finally ready to mathematically define the concepts of a black hole region and an event horizon. Note that for astrophysicists these definitions are of limited use since we do not know for sure how our Universe will look like in the infinite future. However, for proving some theorems that apply to isolated black holes it is useful to introduce these definitions.

Black hole region B. In a globally hyperbolic spacetime[1] M the black hole region B is defined by $B := M - J^-(\mathscr{I}^+)$.

Event horizon H. $H := \dot{J}^-(\mathscr{I}^+) \cap M$. In words: the event horizon of a black hole is given by the boundary of the causal past of future null infinity within some spacetime M. See the Carter–Penrose diagram below (again, wait for next week).



Apparent horizon. Wait for later; qualitatively: expansion of null geodesics either negative or zero, i.e., light-rays cannot "escape". Local definition!

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[1]Globally hyperbolicity can be too strong, e.g. for charged or rotating black holes, which have a Cauchy horizon as inner horizon. In that case the weaker condition of "strong asymptotic predictability" replaces global hyperbolicity, see the beginning of chapter 12 in Wald's book. Strong asymptotic predictability means that no observer outside a black hole can see a singularity.