Slick derivation of Kerr on one page

Flat space in ellipsoidal coordinates

Schwarzschild has spherical symmetry, which is broken to ellipsoidal symmetry by rotation in the Kerr solution. Flat spacetime in ellipsoidal coordinates, $x = \sqrt{r^2+a^2}\,\sin\theta\,\cos\varphi$, $y = \sqrt{r^2+a^2}\,\sin\theta\,\sin\varphi$, $z = r\,\cos\theta$, is given by

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \frac{\Sigma}{r^2+a^2}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 + (r^2+a^2)\,\sin^2\theta\,\mathrm{d}\varphi^2 \tag{1}$$

with $t \in \mathbb{R}$, $r \in \mathbb{R}^+$, $\theta \in [0,\pi)$, $\varphi \sim \varphi + 2\pi$, the definition

$$\Sigma := r^2 + a^2\,\cos^2\theta \tag{2}$$

and some constant parameter a (if $a = 0$ we have manifest spherical symmetry). It is straightforward to check that the metric (1) with (2) has vanishing Riemann tensor and hence describes 4-dimensional Minkowski spacetime.

With hindsight, we represent the line-element above now in so-called Boyer–Lindquist coordinates, which are actually the same coordinates we used already, but reorganized slightly differently.

$$\mathrm{d}s^2 = -\frac{f(r)}{\Sigma}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi\right)^2 + \frac{\Sigma}{f(r)}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\Sigma}\left((r^2+a^2)\,\mathrm{d}\varphi - a\,\mathrm{d}t\right)^2 \tag{3}$$

The function $f(r)$ can be read off from (1).

$$f(r) = r^2 + a^2\,. \tag{4}$$

In the special case $a = 0$ we recover flat space in Schwarzschild coordinates,

$$\mathrm{d}s^2\big|_{a=0} = -\frac{f(r)}{r^2}\,\mathrm{d}t^2 + \frac{r^2}{f(r)}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) \tag{5}$$

since for vanishing a we have $f(r)|_{a=0} = r^2$.

Kerr solution

So far we have just represented flat space in some coordinates that are convenient for our purposes, but we have not introduced any black hole mass. While we do not know how to do this for Kerr, we know already how to do it for Schwarzschild. Namely, take the manifestly spherically symmetric line-element (5) and use for the function $f(r)$ the expression

$$f(r)\big|_{a=0} = r^2 - 2Mr\,. \tag{6}$$

Thus, we know that the metric in Boyer–Lindquist coordinates (3) solves the vacuum Einstein equations $R_{\mu\nu} = 0$ for functions $f(r)$ given by (6) (limit of no rotation, $a = 0$) or (4) (limit of no mass, $M = 0$). A plausible guess for the function $f(r)$ in the presence of rotation *and* mass is thus given by

$$f(r) = r^2 - 2Mr + a^2 \tag{7}$$

since it has the correct limits $M \to 0$ and $a \to 0$. This guess actually works.

It is straightforward (though lengthy) to show that **the metric** (3) **with the function** (7) **solves the vacuum Einstein equations for all values of** a **and** M. This metric is known as the **Kerr solution** (in Boyer–Lindquist coordinates) and describes rotating black holes. The parameter M is called “mass” and a “Kerr parameter”. For vanishing a the Schwarzschild solution (5) with (6) is recovered.

How unique is this solution? Consider a generic Ansatz for the function $f(r)$ in (7), $f(r) = r^2 + a^2 + h(r)$ and require that the metric has vanishing Ricci scalar. The most general such solution is given by $h''(r) = 0$, so that $h(r) = c - 2Mr$, where c and M are constants. The vacuum Einstein equation component $R_{\theta\theta} = 0$ implies $c = 0$. Then also all other components vanish, $R_{\mu\nu} = 0$. Thus, the most general result for $f(r)$ compatible with the vacuum Einstein equations is given by (7), which yields the Kerr solution if plugged into (3).

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