Kaluza–Klein Reduction of Conformally Flat Spaces

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Received 8 September 2006
Communicated by D. V. Ahluwalia-Khalilova

Kaluza–Klein reduction of conformally flat spaces is considered for arbitrary dimensions. The corresponding equations are particularly elegant for the reduction from four to three dimensions. Assuming circular symmetry leads to explicit solutions which also arise from specific two-dimensional dilaton gravity actions.

Keywords: Conformally flat spaces; Kaluza–Klein reduction; two-dimensional dilaton gravity.

1. Introduction

A “conformal tensor” is constructed from the metric tensor \( g_{MN} \) (or Vielbein \( e^A_M \)) and is invariant against Weyl rescaling \( g_{MN} \rightarrow e^{2\sigma} g_{MN} \) (or \( e^A_M \rightarrow e^A\sigma e^A_M \)). Moreover, it vanishes if and only if the space is conformally flat, \( g_{MN} = e^{2\sigma} \eta_{MN} \) (or \( e^A_M = e^A\sigma \delta^A_M \)). In dimension four or greater the conformal tensor is the Weyl tensor. In three dimensions the Weyl tensor vanishes identically, while the Cotton tensor takes the role of probing conformal flatness and Weyl invariance.

Of interest is the behavior of the \( n \)-dimensional conformal tensor under a Kaluza–Klein dimensional reduction to \((n - 1)\) dimensions. Specifically in \( n \) dimensions we take the metric tensor in the form

\[
g_{MN} = e^{2\sigma} \begin{pmatrix} g_{\mu\nu} - a_\mu a_\nu & -a_\mu \\ -a_\nu & -1 \end{pmatrix}.
\]  

Equivalently, for the Vielbein we take

\[
e^A_M = e^{\sigma} \begin{pmatrix} e^{A}_{\mu} & a_\mu \\ 0 & 1 \end{pmatrix}.
\]

2075
Here $g_{\mu\nu}$ is the $(n-1)$-dimensional metric tensor, $e^a_\mu$ is the corresponding Vielbein, $a_\mu$ is an $(n-1)$-dimensional vector. The scalar $\sigma$ plays no role in a conformal tensor and is therefore omitted in the following. It is assumed that the remaining quantities are independent of the $n$th coordinate $x^n$. An $x^n$-independent redefinition of that coordinate acts as a gauge transformation on $a_\mu$. We use capital letters for $n$-dimensional indices, moved by $g_{MN}$, and Greek letters for $(n-1)$-dimensional indices, moved by $g_{\mu\nu}$. The $(n-1)$-dimensional line element

$$ds^2_{(n-1)} = g_{\mu\nu} dx^\mu dx^\nu$$

enters into the $n$-dimensional line element as

$$ds^2_{(n)} = g_{MN} dx^M dx^N = ds^2_{(n-1)} - (a_\mu dx^\mu + dx^n)^2.$$  

Tangent space indices are raised and lowered by the Minkowski metric with signature $(+,\ldots,-)$. All $(n-1)$-dimensional geometric entities are denoted by lower case letters, whereas the $n$-dimensional Riemann tensor and related tensors are denoted by capital letters.

Setting the dimensionally reduced $n$-dimensional conformal tensor to zero determines how our $(n-1)$-dimensional Kaluza–Klein theory fits into an $n$-dimensional, conformally flat space. By exhibiting specific forms for $g_{\mu\nu}$ and $a_\mu$, we provide a classification of conformally flat spaces with (at least) one (conformal) Killing vector.

The above program has been carried out already for the $n=3 \to n=2$ transition, with the Cotton tensor undergoing the reduction. The present work deals with the general case: $n \to n-1$, $n \geq 4$, and also examines in greater detail the $n=4 \to n=3$ reduction, which is especially intriguing since the four-dimensional Weyl tensor produces a vanishing Weyl tensor in $n=3$. For comparison with previous results, we first record some of the formulas of the $n=3 \to n=2$ analysis.

The two-dimensional quantities, which descend from the three-dimensional Cotton tensor with the Kaluza–Klein Ansatz (1) or (2), are expressed in terms of the metric tensor $g_{\mu\nu}$, its Ricci curvature scalar $r$, and the field strength $f$ constructed from $a_\mu$ ($\varepsilon^{01}=1$),

$$\partial_\mu a_\nu - \partial_\nu a_\mu := \sqrt{-g} \varepsilon_{\mu\nu} f.$$  

The equations that ensure vanishing Cotton tensor evaluate the Ricci scalar in terms of $f^2$ and a constant $c$,

$$r = 3f^2 - c$$

(some signs differ from previous papers owing to different conventions; here $R_{MN} := \partial_K \Gamma^K_{MN} - \cdots$). Also, $f$ satisfies a “kink” equation $(d^2 := d_\mu d^\mu)$

$$d^2 f - cf + f^3 = 0$$

(7)
and a traceless equation
\[ d_\mu d_\nu f - \frac{1}{2} g_{\mu\nu} d^2 f = 0. \]  

Here \( d \) is the lower-dimensional covariant derivative. Because the Cotton tensor arises by varying the three-dimensional gravitational Chern–Simons term, the above equations also arise from an action, which is the dimensional reduction of the Chern-Simons term. This action takes the form of a two-dimensional dilaton gravity
\[ I_{\text{CS}} = \frac{1}{8 \pi^2} \int d^2 x \sqrt{-g} (f r - f^3), \]  
except that \( f \) is not a fundamental dilaton field, but is the curl of \( a_\mu \), (5). Equation (6) comes from varying \( I_{\text{CS}} \) with respect to \( a_\mu \), while (7) and (8) are obtained by varying \( g_{\mu\nu} \). A reformulation of (9) as genuine Maxwell-dilaton gravity action is possible as well,
\[ I'_{\text{CS}} = \frac{1}{8 \pi^2} \int d^2 x \sqrt{-g} (X r - Y f + Y X - X^3), \]  
and facilitates the construction of all classical solutions. The dilaton \( X \) is now a fundamental scalar field, which coincides with \( f \) on-shell.

2. Dimensional Reduction from \( n \) to \( (n-1) \) Dimensions

Evaluating the \( n \)-dimensional Riemann tensor \( R^{KLMN} \) on an \( x^n \)-independent metric tensor of the form (1) gives
\[ R^{\mu\nu\lambda\tau} = r^{\mu\nu\lambda\tau} + \frac{1}{4} (f^{\mu\tau} f^{\lambda\nu} - f^{\mu\lambda} f^{\tau\nu} + 2 f^{\mu\nu} f^{\lambda\tau}), \]  
\[ R^{-\lambda\mu\nu} = \frac{1}{2} d^\lambda f^{\mu\nu} - a_\tau R^\tau\lambda\mu\nu, \]  
\[ R^{-\mu-\nu} = -\frac{1}{4} f^{\mu\lambda} f^\lambda\nu - a_\lambda (R^{-\mu\lambda\nu} + R^{-\nu\lambda\mu}) - a_\lambda a_\tau R^{\lambda\mu\tau\nu}. \]  
Here \( R^{\mu\nu\lambda\tau} \) is the \( n \)-dimensional Riemann tensor with all indices evaluated in the \( (n-1) \)-dimensional range; in \( R^{-\lambda\mu\nu} \) the first index refers to the \( n \)th dimension, and this is similarly the case for \( R^{-\mu-\nu} \); \( r^{\mu\nu\lambda\tau} \) is the \( (n-1) \)-dimensional Riemann tensor and
\[ f_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu. \]  

The \( n \)-dimensional Ricci tensor has components
\[ R^{\mu\nu} = r^{\mu\nu} - \frac{1}{2} f^{\mu\lambda} f^\lambda\nu, \]  
\[ R^{-\mu} = -\frac{1}{2} d_\nu f^{\nu\mu} - a_\nu R^{\mu\nu}. \]
\[ R^{-F} = -\frac{1}{4} f^{\mu\nu} f_{\nu\mu} - 2a_\mu R^{-\mu} - a_\mu a_\nu R^{\mu\nu}. \] (13c)

Finally, the Ricci scalar reduces as

\[ R = r - \frac{1}{4} f^{\mu\nu} f_{\nu\mu}. \] (14)

Of course (13) and (14) follow from (11) by taking the appropriate traces. These formulas are equivalent to the Gauss–Codazzi equations, which, however, are usually presented in a different manner.

Next we employ the formula that expresses the Weyl tensor \( C^{KLMN} \) in terms of the Riemann tensor and its traces in \( n \) dimensions

\[ C^{KLMN} := R^{KLMN} - \frac{2}{n-2} (g^{K[M} S^{N]L} - g^{L[M} S^{N]K}), \] (15a)

where \( g^{K[M} S^{N]L} := (g^{KM} S^{NL} - g^{KN} S^{ML})/2 \) denotes antisymmetrization and

\[ S^{NL} := R^{NL} - \frac{1}{2(n-1)} g^{NL} R \] (15b)

is the Schouten tensor. A similar formula holds in \( (n-1) \) dimensions, with \( n-1 \) replacing \( n \). We express the various curvatures in (15) in terms of the \( (n-1) \) dimensional expressions in (11)–(14), and then reexpress \( r^{\mu\nu\lambda\tau} \) in terms of the \( (n-1) \) dimensional Weyl tensor, using (15) with \( n \) replaced by \( n-1 \). This gives our principal result of this section: the relation between the \( n \)- and \( (n-1) \) dimensional Weyl tensors when the metric tensor is represented as in (1)

\[ C^{\mu\nu\lambda\tau} = e^{\mu\nu\lambda\tau} + \frac{2}{n-3} (g^{\mu[\lambda} e^{\tau]\nu} - g^{\nu[\lambda} e^{\tau]\mu}) \]

\[ + \frac{1}{4} (f^{\mu\tau} f^{\lambda\nu} - f^{\mu\lambda} f^{\tau\nu} + 2f^{\mu\nu} f^{\lambda\tau}) \]

\[ + \frac{3}{2(n-3)} (g^{\mu[\lambda} t^{\tau]\nu} - g^{\nu[\lambda} t^{\tau]\mu}), \] (16a)

where

\[ e^{\mu\nu} := \frac{1}{n-2} \left( r^{\mu\nu} - \frac{1}{n-1} g^{\mu\nu} r - \frac{n}{4} \left( f^{\mu\lambda} f^{\lambda\nu} - \frac{1}{n-1} g^{\mu\nu} f^{\lambda\tau} f^{\tau\lambda} \right) \right), \] (16b)

\[ t^{\mu\nu} := f^{\mu\lambda} f^{\lambda\nu} - \frac{1}{2(n-2)} g^{\mu\nu} f^{\lambda\nu} f_{\tau\lambda}. \] (16c)

Note that \( e^{\mu\nu} = C^{\mu\nu} \lambda \nu \) and \( e^{\mu}_{\mu} = 0 \) (because only the traceless part of the Ricci tensor and a traceless combination of field strengths enter \( e^{\mu\nu} \)). The remaining independent component is

\[ C^{-\lambda\mu\nu} = \frac{1}{2} d^{\lambda} f^{\mu\nu} + \frac{1}{(n-2)} g^{\lambda[\mu} d^{\nu]\tau} - a_{\tau} C^{\tau\lambda\mu\nu}. \] (16d)

The further quantity \( C^{-\mu\nu} \) is determined by the previous equation, because \( C^{KLMN} \) is traceless in all paired indices.
Our formulas (11)–(16) may also be presented with tangent space components, after contraction with Vielbeine. The only change is that the gauge-dependent, $a_{\mu}$-dependent contributions are absent.

The result (16) obviously makes sense only for $n \geq 4$; at $n = 4$, $c^{\mu \nu \lambda \tau}$ — the three-dimensional Weyl tensor — vanishes. Other simplifications occur as well, and this will be described below.

3. Equations for Conformal Flatness

3.1. General $n \geq 4$

Demanding that the $n$-dimensional Weyl tensor (16) vanish requires that $C^{\mu \nu \lambda \tau}$ and $C_{\lambda \mu \nu}$ vanish. Thus an $n$-dimensional conformally flat space obeys after a Kaluza–Klein reduction the following equations in $(n-1)$ dimensions:

\[
\begin{align*}
    c^{\mu \nu \lambda \tau} + \frac{1}{4} \left( f_{\mu \tau} f^{\lambda \nu} - f_{\mu \lambda} f^{\tau \nu} + 2 f^{\mu \nu} f_{\lambda \tau} \right) + \frac{3}{2(n-3)} \left( g^{\nu(\lambda \tau)} - g^{\nu(\lambda \tau)} \right) &= 0, \\
    c^{\mu \nu} &= 0 \
\rightarrow \quad r^{\mu \nu} - \frac{1}{n-1} g^{\mu \nu} r = \frac{n}{4} \left( f^{\mu \tau} f_{\lambda \nu} - \frac{1}{n-1} g^{\mu \nu} f_{\lambda \tau} f_{\lambda \tau} \right), \\
    d^\lambda f^{\mu \nu} + \frac{2}{n-2} g^{\lambda \mu} d_r f^{\nu \tau} &= 0.
\end{align*}
\]  

These equations are traceless in all paired indices; the last one is consistent with the gauge theoretic Bianchi identity.

A general result consequent to (17b) and (17c) expresses the scalar curvature $r$ in terms of $f^{\mu \nu} f_{\nu \mu}$. This is established in the following manner. Take the covariant divergence of $c^{\mu \nu}$ and use the Bianchi identity $d^\mu r_{\mu \nu} = \frac{1}{2} \partial_\nu r$. This gives

\[
\partial_\nu \left( 2(n-3)r + nf^{\lambda \tau} f_{\tau \lambda} \right) = n(n-1)(d_\mu f^{\mu \lambda} f_{\lambda \nu} + f^{\mu \lambda} d_\mu f_{\lambda \nu}).
\]  

The last term is evaluated from (17c) as

\[ f^{\mu \lambda} d_\mu f_{\lambda \nu} = \frac{1}{n-2} d_\mu f^{\mu \lambda} f_{\lambda \nu} \]  

and combines with the first term on the right-hand side of (18a) to give

\[
\frac{n(n-1)^2}{n-2} d_\mu f^{\mu \lambda} f_{\lambda \nu}. \]  

This is now shown to be a total derivative. To accomplish the result, begin with the equality that follows from the Bianchi identity for $f_{\lambda \nu}$

\[
f^{\mu \lambda} d_\mu f_{\lambda \nu} = -f^{\mu \lambda} (d_\lambda f_{\nu \mu} + d_\nu f_{\mu \lambda}) = -f^{\mu \lambda} d_\mu f_{\lambda \nu} + \frac{1}{2} \partial_\nu (f^{\mu \lambda} f_{\lambda \mu}).
\]  

Equivalently

\[ f^{\mu \lambda} d_\mu f_{\lambda \nu} = \frac{1}{4} \partial_\nu (f^{\mu \lambda} f_{\lambda \mu}). \]
But from (18b), this implies
\[ d_\mu f^{\mu \lambda} f_{\lambda \nu} = \frac{n-2}{4} \partial_\nu (f^{\mu \lambda} f_{\lambda \mu}). \]
(18e)

Therefore, the right-hand side of (18a) is indeed a total derivative, and its integration finally gives
\[ r = \frac{n(n+1)}{8} f^{\mu \nu} f_{\nu \mu} - c, \]
(19)
where \( c \) is a constant. This is a generalization to arbitrary dimension of our \( n = 3 \) result (6).

A further general, but trivial result may be derived. Equations (17a) are solved by vanishing \( f_{\mu \nu} \) and pure gauge \( a_\mu \), because then (17c) is true and (17b) shows that the space is maximally symmetric. Therefore the Weyl tensor vanishes, thereby satisfying (17a). Finally, Eq. (19) consistently identifies the scalar curvature with a constant.

If viewed as a field equation, (17b) has a status comparable to the Einstein equation as it connects geometry (left-hand side) with matter (right-hand side). Incidentally, if \( n = 5 \), the right-hand side is proportional to the energy–momentum tensor of a four-dimensional Maxwell field, concurrent with the conformal properties of the latter in four dimensions.

3.2. \( n = 4 \)

Our equations simplify dramatically when \( n = 4 \) and the reduced system is three-dimensional. First of all, the \( n = 3 \) Weyl tensor vanishes identically. Moreover, the field strength \( f_{\mu \nu} \) may be presented in terms of its dual \( f^\mu \)
\[ f_{\mu \nu} = \sqrt{g} \varepsilon_{\mu \nu \lambda \tau} f^\lambda, \quad d_\mu f^\mu = 0. \]
(20)

Substituting (20) into (16a) shows that the last two \( f \)-dependent quantities vanish identically, leaving
\[ C^{\mu \nu \lambda \tau} = 2(g^{\mu [\lambda} c^{\tau ] \nu} - g^{\nu [\lambda} c^{\tau ] \mu}), \]
(21a)
\[ c^{\mu \nu} = \frac{1}{2} \left( r_{\mu \nu} - \frac{1}{3} g_{\mu \nu} r - f^\mu f^\nu + \frac{1}{3} g^{\mu \nu} f^2 \right) \]
(21b)
\[ (f^2 := f^\mu f_\mu), \quad \text{while (16d) becomes} \]
\[ C^{-\lambda \mu \nu} = \frac{\varepsilon^{\mu \nu \tau}}{2 \sqrt{g}} (d^\lambda f_\tau + d_\tau f^\lambda) - a_\tau C^{\tau \lambda \mu \nu}. \]
(21c)

Therefore, the vanishing of the four-dimensional Weyl tensor when reduced to three dimensions requires according to (21a) and (21b) that \( c^{\mu \nu} \) vanish
\[ r_{\mu \nu} - \frac{1}{3} g_{\mu \nu} r = f^\mu f^\nu - \frac{1}{3} g^{\mu \nu} f^2, \]
(22a)
while according to (21c) $f^\mu$ when it is non-vanishing is a Killing vector of the three-dimensional geometry

$$d_\mu f_\nu + d_\nu f_\mu = f^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu} \partial_\mu f^\lambda + g_{\mu\lambda} \partial_\nu f^\lambda = 0. \quad (22b)$$

The $n = 4$ restriction of (19) is a straightforward consequence of Eqs. (22),

$$r = -5f^2 - c. \quad (23)$$

Equation (22a) may be presented in “Einstein form”

$$r_{\mu\nu} - \frac{1}{2} g_{\mu\nu} r = f_\mu f_\nu + \frac{1}{6} g_{\mu\nu} (c + 3f^2). \quad (24)$$

For time-like $f^\mu$ this has a hydrodynamical interpretation with pressure $P = -\frac{c}{6} - \frac{1}{2} f^2$ and energy density $E = \frac{3}{2} f^2 + \frac{c}{6}$.

When $f^\mu$ vanishes, (22a) shows that our three-dimensional space is maximally symmetric. This gives the line element

$$ds^2(4) = ds^2(3) - dz^2, \quad (25a)$$

where $z = x^4$ and $ds^2(3)$ describes Minkowski or $(A)dS_3$ space,

$$ds^2(3) = (1 \mp \lambda^2 \rho^2)^{-1} dt^2 - (1 \mp \lambda^2 \rho^2)^{-1} d\rho^2 - \rho^2 d\theta^2. \quad (25b)$$

The upper (lower) sign is valid for $(A)dS_3$. Minkowski space is obtained if the parameter $\lambda$ vanishes. The Ricci scalar is given by $r = \mp 6\lambda^2$. Of course the four-dimensional line element (25a) is conformally flat by construction.

For non-vanishing $f^\mu$, the right-hand side of (22a) must not vanish, for otherwise the metric would be singular. Therefore, with non-vanishing $f^\mu$, the space is not maximally symmetric. In that case a second Killing vector exists if $d_\mu f_\nu$ does not vanish identically. This statement can be derived as follows. The relation $d_\mu d_\nu f_\lambda = R_{\lambda\mu\nu\tau} f^\tau$ — upon expressing the Riemann tensor in terms of the Ricci tensor and the latter in terms of the Killing vector using (22a) and (23) — establishes

$$d_\mu d_\nu f_\lambda = \frac{1}{6} (g_{\mu\nu} f_\lambda - g_{\mu\lambda} f_\nu) (3f^2 + c). \quad (26a)$$

Defining

$$F^\mu := \frac{\varepsilon^{\mu\nu\lambda}}{\sqrt{g}} d_\nu f_\lambda \quad (26b)$$

leads by virtue of (26a) to the Killing equation for $F^\mu$,

$$d_\mu F_\nu + d_\nu F_\mu = 0. \quad (26c)$$

We say that $F^\mu$ is a Killing vector dual to the Killing vector $f^\mu$, which enters our system of equations. It then remains a further problem whether this dual Killing vector determines an (additional) geometry from (22). (The dual Killing vector $F^\mu$ must not be confused with the gauge theoretic dual of $f^\mu$, viz. $f_\mu \nu = \sqrt{g} \varepsilon_{\mu\nu\lambda} f^\lambda$.)

If $d_\mu f_\nu = 0$ (but $f^\mu \neq 0$), then there is no dual Killing vector of the form (26b). From (26a), one may deduce $3f^2 + c = 0$, which together with (23) leads to constant
curvature $r = -2f^2$. The Killing condition (22b) is fulfilled trivially and (22a) or (24) simplifies to

$$r_{\mu\nu} - \frac{1}{2}g_{\mu\nu}r = f_\mu f_\nu.$$  \hspace{1cm} (27)

(In a hydrodynamical interpretation this is the Einstein equation with a pressureless perfect fluid source of constant energy density if $f^\mu$ is time-like.) We shall not analyze this case further here but encounter it again in Sec. 4.1 below.

While our Eqs. (22) possess an undeniable elegance, we have not succeeded in analyzing them further in full generality. Due to the presence of a Killing vector $f^\mu$ one can perform a second Kaluza–Klein split (1) for the three-dimensional metric and study the ensuing equations. In the next section we present all solutions with vanishing vector potential (with respect to the second Kaluza–Klein split), i.e., the three-dimensional Killing vector is required to be hypersurface orthogonal and the metric tensor is block diagonal. With the notable exception of these solutions, we have not found an action that would generate (22), since the four-dimensional Weyl tensor is not known to be the variation of any action.

4. Circularly Symmetric Solutions

When $n > 4$, Eqs. (17) determining the geometry and the field strength are daunting, and we have not attempted to solve them. For $n = 4$ we have not found the general solution to (22), but we have constructed special solutions, based on an Ansatz for the three-dimensional line element similar to (1),

$$ds^2(3) = g_{\alpha\beta}dx^\alpha dx^\beta - \phi^2 d\theta^2$$  \hspace{1cm} (28)

but with vanishing vector field for simplicity. The two-dimensional metric $g_{\alpha\beta}$ and the scalar field $\phi$ are required to depend solely on the two-dimensional coordinates $x^\alpha$, so we have circular symmetry due to the Killing vector $(\partial_\theta)^\mu$. A motivation for this Ansatz is that it encompasses the circularly symmetric case $\phi = \rho$, $x^\alpha = (t, \rho)$. The three-dimensional Riemann and Ricci tensors, as well as the Ricci scalar, can be related to intrinsically two-dimensional quantities, which are denoted in capital letters, with superscript (2).

Riemann tensor:

$$\begin{align*}
R^{\alpha\beta\gamma\delta} &= (2)R^{\alpha\beta\gamma\delta}, \\
R^{\theta\alpha\beta\gamma} &= 0, \\
R^{\theta\alpha\theta\beta} &= \frac{1}{\phi^2}(\nabla^\beta \nabla^\alpha \ln \phi + (\nabla^\alpha \ln \phi)(\nabla^\beta \ln \phi)),
\end{align*}$$  \hspace{1cm} (29a-c)

Ricci tensor:

$$\begin{align*}
R^\alpha{}_{\beta} &= (2)R^\alpha{}_{\beta} - \nabla^\alpha \nabla^\beta \ln \phi - (\nabla^\alpha \ln \phi)(\nabla^\beta \ln \phi), \\
R^\alpha{}_{\theta} &= 0, \\
R^\theta{}_{\theta} &= \frac{1}{\phi^2}(\Box \ln \phi + (\nabla \ln \phi)^2),
\end{align*}$$  \hspace{1cm} (29d-f)
Ricci scalar:
\[ r = (2)R - 2(\Box \ln \phi + (\nabla \ln \phi)^2). \] (29g)

The indices \( \alpha, \beta, \gamma \), and \( \delta \) from the beginning of the Greek alphabet range over \((0, 1)\), and \( \theta \) denotes the Killing coordinate. The two-dimensional covariant derivative is denoted by \( \nabla_\alpha \) and \( \Box := \nabla_\alpha \nabla_\alpha \).

We seek now solutions to (22) with a line element of the form (28). First we check the Killing condition (22b) which ramifies into
\[ \nabla_\alpha (f_\beta) = 0 \rightarrow f_\lambda \partial_\lambda g_{\alpha\beta} + g_{\lambda\beta} \partial_\alpha f_\lambda + g_{\lambda\alpha} \partial_\beta f_\lambda = 0, \] (30a)
\[ \nabla_\alpha (f_\theta) = 0 \rightarrow -\phi^2 \partial_\alpha f^\theta + g_{\alpha\beta} \partial_\theta f^\beta = 0, \] (30b)
\[ \nabla_\theta f_\theta = 0 \rightarrow \partial_\theta f^\theta + f^\alpha \partial_\alpha \ln \phi = 0. \] (30c)

Next we take note of the condition (22a) that \( c^{\mu\nu} \) vanish. The five independent components split into the \( \alpha\theta \)-part
\[ f^\alpha f^\theta = 0 \] (31a)
and the \( \alpha\beta \)-part
\[ \epsilon^{\alpha\beta} - \frac{1}{3} g^{\alpha\beta} r = f^\alpha f^\beta - \frac{1}{3} g^{\alpha\beta} (f^\gamma f_\gamma + f^\theta f_\theta). \] (31b)

The \( \theta\theta \)-part is redundant because \( c^{\mu\nu} \) is traceless. Equation (31a) requires either \( f^\theta = 0 \), so the Killing vector would be intrinsically two-dimensional, or \( f^\alpha = 0 \), so the Killing vector would have no two-dimensional component at all. A mixing, \( f^\theta \neq 0 \neq f^\alpha \), is not possible. So if \( f^\alpha \neq 0 \) — and thus \( f^\theta = 0 \) — we deduce from (30b) that \( f^\alpha \) must be independent of \( \theta \) and from (30c) that it must be orthogonal to \( \partial_\alpha \phi \). On the other hand, if \( f^\theta \neq 0 \) — and thus \( f^\alpha = 0 \) — we deduce from (30b) and (30c) that \( f^\theta \) must be constant. Thus, the three-dimensional Killing vector always must be \( \theta \)-independent.

4.1. Solutions of constant \( \phi \)

Before studying the general problem we shall focus on the much simpler case of constant \( \phi \), scaled to unity, whence \( r = (2)R \) and \( r_{\alpha\beta} = (2)R_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} (2)R \). For \( f^\theta = 0 \) and \( f^\alpha \neq 0 \), there is no solution to (31b). For \( f^\alpha = 0 \) and \( f^\theta = \lambda = \text{constant} \), all conditions (30) and (31) are satisfied provided the Ricci scalar obeys
\[ r = (2)R = -2f^\theta f_\theta = 2\lambda^2 \geq 0. \] (32)

In our notation this implies \( \text{AdS}_2 \) (or the two-dimensional Minkowski space if the inequality is saturated). It should be noted that the constant \( c \) in (23) may not be chosen freely but rather is determined as \( c = 3\lambda^2 \).
Thus all solutions with $\phi$ in (28) constant are either the two-dimensional Minkowski space or $\text{AdS}_2$ and therefore admit three Killing vectors in two-dimensional which may be lifted to three-dimensional ones, thereby supplementing the $f^\mu$ with which we have started. The three-dimensional line element
\[
\text{ds}_{(3)}^2 = (1 + \lambda^2 \rho^2) dt^2 - (1 + \lambda^2 \rho^2)^{-1} d\rho^2 - d\theta^2
\] (33a)
enters the four-dimensional one (using the gauge $a_t = -\lambda \rho$ and $a_\rho = a_\theta = 0$)
\[
\text{ds}_{(4)}^2 = \text{ds}_{(3)}^2 - (\lambda \rho dt - dz)^2,
\] (33b)
which again is conformally flat by construction. All four Killing vectors can be lifted to four dimensions (only in one case this is not entirely trivial as the corresponding Killing vector acquires a $z$-component).

Note that the dual Killing vector from (26b) vanishes identically. This brings us back to the case $d_\mu f_\nu = 0$ mentioned around (27). What we have shown above is that for the Ansatz (28) and constant $\phi$ the special case $d_\mu f_\nu = 0$ emerges. To show that the converse holds (for space-like $f^\mu$) without loss of generality we can make a Kaluza–Klein Ansatz
\[
\text{ds}_{(3)}^2 = g_{\alpha\beta} dx^\alpha dx^\beta = e^{2\hat{\sigma}} (\hat{g}_{\alpha\beta} dx^\alpha dx^\beta - (\hat{a}_\alpha dx^\alpha + d\theta)^2),
\] (34a)
where the coordinate $\theta$ corresponds to the Killing direction implied by $f^\mu$ and $\alpha, \beta$ range from 0 to 1. Therefore, $f^\alpha = 0$ and $f^\theta = \text{constant}$. All quantities depend on $x^\alpha$ only. So far we have just exploited the fact that $f^\mu$ is a (space-like) Killing vector and therefore the metric may be brought into the adapted form (34a); now we employ the property that $f^\mu$ is covariantly constant. As a consequence $\partial_\alpha g_{\theta\theta} = 0$, so the scalar field $\hat{\sigma}$ must be constant, and $e^{2\hat{\sigma}}$ can be scaled to 1. Next, we evaluate (27) with upper indices for the $\alpha\beta$-part and obtain by virtue of (13a) and (14) (now evaluated for the reduction from 3 $\rightarrow$ 2)
\[
(2)R^{\alpha\beta} - \frac{1}{2} f^{\alpha\gamma} f^{\gamma\beta} - \frac{1}{2} \hat{g}^{\alpha\beta} \left( (2)R - \frac{1}{4} f^{\gamma\delta} f^{\delta\gamma} \right) = f^\alpha f^\beta = 0,
\] (34b)
where $\hat{f}_{\alpha\beta} := \partial_\alpha \hat{a}_\beta - \partial_\beta \hat{a}_\alpha$. With the identity $(2)R^{\alpha\beta} = \frac{1}{2} \hat{g}^{\alpha\beta} (2)R$ and the definition $\sqrt{-g} e_{\alpha\beta} f := \hat{f}_{\alpha\beta}$, Eq. (34b) simplifies to
\[
\hat{g}^{\alpha\beta} \hat{f}^2 = 0.
\] (34c)
The only solution to (34c) with non-degenerate two-dimensional metric $\hat{g}^{\alpha\beta}$ is given by $\hat{f} = 0$, so $\hat{a}_\alpha$ must be pure gauge. Thus, the two-dimensional vector potential $\hat{a}_\alpha$ may be chosen to vanish, leading to (28) with $\phi = 1$. We have noted above that all such solutions exhibit three Killing vectors in addition to $f^\mu$.

### 4.2. Solutions of non-constant $\phi$

According to the previous analysis, there are two cases to consider for the Killing vector $f^\mu = (f^\alpha, f^\theta)$ in our system of equations (30) and (31): intrinsically
two-dimensional Killing vector \((f^a \neq 0, f^\theta = 0)\) or Killing vector with no two-dimensional component \((f^a = 0, f^\theta = \lambda = \text{constant})\). We discuss each case in turn.

4.2.1. **Intrinsically two-dimensional Killing vector** \((f^a \neq 0, f^\theta = 0)\)

We may always bring the two-dimensional portion of the line element (28) into the Eddington–Finkelstein form

\[
g_{\alpha\beta}dx^\alpha dx^\beta = e^{Q(X)}(2du dX + K(X)du^2).
\] (35)

The metric functions depend on only one variable because, by hypothesis, there exists the Killing vector \(f^a\), which according to (35) is proportional to \((\partial_u)^a\). In components \(f^a = (f^u = 1, f^X = 0)\), where no generality is lost by rescaling the constant \(f^u\) to 1. The Killing conditions (30) require that \(\phi\) is a function of \(X\) only. The presence of the prefactor \(e^{Q(X)}\) in (35) still allows arbitrariness in the choice of the \(X\) coordinate, which we shall fix by setting \(\phi^2 = X\). Equation (31a) is obviously satisfied, and the remaining three conditions encoded in (31b) simplify to the following two:

\[
e^{2Q} = \frac{1}{4X^2} + \frac{Q'}{2X},
\] (36a)

\[
K'' + K'\left(\frac{Q'}{2X} - \frac{1}{2X}\right) + K\left(Q'' - \frac{1}{2X}Q'\right) = 0.
\] (36b)

To solve (36a), differentiate that equation and eliminate the exponential, leaving a differential equation for \(Q'\) which is easily solved as

\[
Q' = \frac{1}{2X} - \frac{1}{2(X - a)}, \quad a \in \mathbb{R}.
\] (37a)

We assume \(0 \leq X \leq a\) and rescale \(X\) by \(x = X/a\). Thus \(e^{-Q}\) is proportional to \(\sqrt{x} \sqrt{1-x}\). The proportionality constant is irrelevant and may be absorbed into a global redefinition of units of length. Therefore we set it to 1.

\[
e^{-Q} = \sqrt{x} \sqrt{1-x}.
\] (37b)

The solution to (36b) then becomes

\[
K = e^{-Q}\left(A + B\sqrt{1-x}\right), \quad A, B \in \mathbb{R},
\] (37c)

where \(A, B\) are integration constants. With

\[
\Phi = 2\arcsin \sqrt{x} \quad (0 \leq \Phi \leq \pi)
\] (38)

the three-dimensional line element reads

\[
ds^2_{(3)} = 2dud\Phi + du^2(A + B \cos(\Phi/2)) - a \sin^2(\Phi/2)d\theta^2.
\] (39a)

The coordinate redefinition

\[
tanh(\tilde{\Phi}/2) = \sin(\Phi/2)
\] (39b)
allows an alternative presentation of the line element (39a) as
\[ ds^2_{(3)} = \frac{1}{\cosh(\tilde{\Phi}/2)}(2dud\tilde{\Phi} + du^2(B + A \cosh(\tilde{\Phi}/2))) - a \tanh^2(\tilde{\Phi}/2)d\theta^2. \] (39c)

Note that in a sense $A$ and $B$ have interchanged their roles. This will be elaborated in the next section. The corresponding Ricci scalar,
\[ r = -\frac{5B}{4} \cos(\Phi/2) - \frac{A}{2} = -\frac{5B}{4} \frac{1}{\cosh(\tilde{\Phi}/2)} - \frac{A}{2}, \] (39d)

has no singularities. Upon redefining $t = u + h(\Phi)$, $h(\Phi) = 1/(A + B \cos(\Phi/2))$, and $\rho = \sqrt{a}\sin(\Phi/2)$, the line element
\[ ds^2_{(3)} = (A + B \sqrt{1 - \rho^2/a})dt^2 - \frac{4/a}{(1 - \rho^2/a)}(A + B \sqrt{1 - \rho^2/a})^{-1}d\rho^2 - \rho^2 d\theta^2 \] (39e)

is in a conventional static and circularly symmetric form, and the relevant Killing vector is time-like ($f_t = 1$, $f_\rho = f_\theta = 0$). The four-dimensional line element (in the gauge $a_t = a_\rho = 0$ and $a_\theta = -2\sqrt{a - \rho^2}$)
\[ ds^2_{(4)} = ds^2_{(3)} - (2\sqrt{a - \rho^2}d\theta - dz)^2 \] (39f)

is again conformally flat by construction. The constant $a$ can be eliminated from (39f) at the expense of redefining $\rho \to \rho/\sqrt{a}$ and $\theta \to \theta/\sqrt{a}$.

Note that the Killing vector $F^\mu$ dual to our $f^\mu : (f^u = 1, f^\rho = 0, f^\theta = 0)$ is given by
\[ F^\mu = \varepsilon^{\mu\nu\lambda\kappa} \partial_\nu(g_{\lambda\kappa}f^\tau) = \frac{\varepsilon^{\mu\phi\alpha}}{\sqrt{g}} \partial_\phi g_{uu}. \] (40)

Evidently only the $\theta$ component survives in $F^\mu$, and it is a constant. This leads to the other solution to our problem, with $f^\alpha = 0$, $f^\theta = \text{constant} \neq 0$.

4.2.2. *Intrinsically two-dimensional dual Killing vector* ($f^\alpha = 0$, $f^\theta \neq 0$)

Upon choosing the constant $f^\theta$ to be 1, we find that the dual Killing vector
\[ F^\mu = \varepsilon^{\mu\nu\lambda\alpha} \partial_\nu(g_{\lambda\alpha}f^\tau) = \frac{\varepsilon^{\mu\alpha\theta}}{\sqrt{g}} \partial_\alpha \phi^2 \] (41)

possesses only $\alpha$ components, i.e., is an intrinsically two-dimensional Killing vector. Without loss of generality we apply again the Eddington–Finkelstein form (35) for the two-dimensional line element. Further, we now fix $X = \phi$. 
Again the condition (31b) leads to two independent equations that read

\[ Q' = 0, \tag{42a} \]
\[ K'' - K' \left( \frac{Q'}{X} + 1 \right) = 2e^Q X^2 \tag{42b} \]

with solutions

\[ e^Q = 1, \tag{43a} \]
\[ K = \frac{1}{4} X^4 + AX^2 + B \quad (A, B \in \mathbb{R}), \tag{43b} \]

where \( A \) and \( B \) are integration constants. The integration constant inherent to \( Q \) again has been fixed conveniently and without loss of generality. Therefore, the three-dimensional line element becomes

\[ ds^2 = 2dudX + \left( \frac{1}{4} X^4 + AX^2 + B \right) du^2 - X^2 d\theta^2. \tag{44a} \]

The corresponding Ricci scalar,

\[ r = 5X^2 + 6A, \tag{44b} \]

is singular for \( |X| \to \infty \). With the coordinate transformation \( t = u + h(X), \quad h' = 1/K \), and \( \rho = X \) the three-dimensional line element

\[ ds^2 = \left( \frac{1}{4} \rho^4 + A\rho^2 + B \right) dt^2 - \left( \frac{1}{4} \rho^4 + A\rho^2 + B \right)^{-1} d\rho^2 - \rho^2 d\theta^2 \tag{44c} \]

is again in a conventional static and circularly symmetric form and the relevant Killing vector is space-like \((f^t = f^\rho = 0, \ f^\theta = 1)\). The four-dimensional line element (in the gauge \( a_\theta = a_\rho = 0 \) and \( a_t = -\rho^2/2 \))

\[ ds^2 = ds^2 - \left( \frac{1}{2} \rho^2 dt - dz \right)^2 \tag{44d} \]

is of course again conformally flat by construction.

Finally we observe that the Killing vector dual to our \( f^\mu : (f^\alpha = 0, f^\theta = 1) \) possesses only a \( u \) component and also is constant. In this way we are brought back to the previous case: intrinsically two-dimensional Killing vector.

5. Two-Dimensional Dilaton Gravity

One may wonder whether the two-dimensional part of the solutions (39a) and (44a) can be derived from some action principle. The purpose of this section is to show that the answer is affirmative. Let us collect the evidence obtained so far: the field content of our three-dimensional theory comprises the two-dimensional metric \( g_{\alpha\beta} \) and a scalar field \( \phi \); one obtains maximally symmetric two-dimensional spaces if the scalar field is constant; there is always (at least) one two-dimensional Killing vector. Therefore, we may conjecture that there should be an effective description
in terms of two-dimensional dilaton gravity because it shares all these features.\(^3\) Thus, we would like to investigate now whether the conditions (22) with the \textit{Ansatz} (28) could follow as equations of motion from an action

\[ I = \frac{1}{8\pi^2} \int d^2x \sqrt{-g} (XR + U(X)(\nabla X)^2 - V(X)), \]  

(45)

where the dilaton field \( X \) is some function of \( \phi \) and \( U, V \) are arbitrary functions defining the model. Since there is no obvious way to obtain an action in four dimensions leading to \( C_{KLMN} = 0 \) as equations of motion, such a construction is of particular interest. We have dropped the superscript \((2)\) of the two-dimensional Ricci scalar \( R \) in (45) as from now on we work almost exclusively in two dimensions and no confusion with four-dimensional quantities should arise.

We recall now a basic result of two-dimensional dilaton gravity\(^3\): The generic solution to the equations of motion following from (45) is parameterized by one constant of motion, \( M \), and using the dilaton field \( X \) as one of the coordinates leads to the line element (35) with

\[ Q(X) = -\int_{\Phi}^{X} U(y) dy, \]

\[ K(X) = \int_{\Phi}^{X} e^{Q(y)} V(y) dy. \]

(46a)

The ambiguities in defining \( Q \) and \( K \) due to the integration constants from the lower limits correspond to a simple coordinate redefinition of \( u \) and an arbitrary aforementioned constant \( M \) in the solution (a “modulus”), respectively. [In addition there may be isolated solutions (in the sense of not possessing a modulus), so-called constant dilaton vacua, which have a constant dilaton \( X \) solving the equation \( V(X) = 0 \) and leading to maximally symmetric spaces with curvature \( R = V'(X) \).]

5.1. \textit{Intrinsically two-dimensional Killing vector}

The two-dimensional line element extracted from (39a), with \( \Phi \) replaced by \( X \),

\[ ds^2 = 2dudX + du^2(A + B \cos(X/2)) \]  

(47a)

follows from the two-dimensional dilaton gravity action

\[ I_1 = \frac{1}{8\pi^2} \int d^2x \sqrt{-g}(XR + \frac{B}{2} \sin(X/2)), \]  

(47b)

where \( B \) is a parameter of the action and \( A \) emerges as a constant of motion. So comparison with (45) establishes the potentials

\[ U(X) = 0, \quad V(X) = -\frac{B}{2} \sin(X/2). \]  

(47c)

The alternative presentation (39c), with \( \tilde{\Phi} \) replaced by \( \tilde{X} \), gives the line element

\[ ds^2 = \frac{1}{\cosh(X/2)}(2dud\tilde{X} + du^2(B + A \cosh(\tilde{X}/2))) \]  

(48a)
with the roles of $A, B$ interchanged. Indeed, there is an alternative action

$$I_1 = \frac{1}{8\pi^2} \int d^2x \sqrt{-g} \left( \hat{X} R + \frac{1}{2} \tanh(\hat{X}/2)(\nabla \hat{X})^2 - \frac{A}{4} \sinh \hat{X} \right),$$

(48b)
depending parametrically on $A$ with $B$ emerging as a constant of motion. The potentials are given by

$$\hat{U}(\hat{X}) = \frac{1}{2} \tanh(\hat{X}/2), \quad \hat{V}(\hat{X}) = \frac{A}{4} \sinh \hat{X}.$$  

(48c)

Instead of a “sine-Gordon” potential, (48b) exhibits not only a “sinh-Gordon” potential but also a kinetic term for the field $\hat{X}$. This formulation seems superior from a physical point of view for three reasons: (i) $\hat{X} \in [0, \infty)$ has non-compact support. (ii) The constant of motion $B$ really plays the physical role of the ADM mass (cf. Sec. 5.1 in Ref. 3). (iii) The ground state solution $B = 0$ is Minkowski space; actually, this is a consequence of the “Minkowskian ground state property” property $Ve^{2Q} \propto \hat{U}$ [cf. (3.40) in Ref. 3].

The global structure of (48a) is very similar to the one of the Schwarzschild black hole: for $\hat{X} \to \infty$, geometry is flat, and for $B = 0$, a Killing horizon emerges. However, there is no curvature singularity at $\hat{X} = 0$. For small values of the dilaton both actions asymptote to the Jackiw–Teitelboim model.\(^{4,5}\) Besides these features nothing noteworthy can be remarked about these geometries; there is no “kink”-like structure, which was found in the $n = 3 \to n = 2$ transition.

### 5.2. Intrinsically two-dimensional dual Killing vector

The two-dimensional line element extracted from (44a)

$$ds^2 = 2dudX + \left( \frac{1}{4} X^4 - \frac{Y}{2} X^2 - 2M \right) du^2$$

(49a)

follows from the two-dimensional dilaton gravity action

$$I_2 = \frac{1}{8\pi^2} \int d^2x \sqrt{-g} \left( XR + YX - X^3 \right).$$

(49b)

[Here (49a) is presented with $A$ and $B$ replaced by $-Y/2$ and $-2M$, respectively.] Remarkably, this is almost identical to (10) in the following sense: upon integrating out the vector field $a_\mu$, the scalar field $Y$ in (10) evidently is constant and may be chosen to coincide with the parameter $Y$ in (49b). The potentials are given by

$$U(X) = 0, \quad V(X) = YX - X^3.$$  

(49c)

The global structure of the geometry (49a) has been analyzed already\(^{2}\); depending on the signs and magnitudes of $Y$ and $M$ there may be up to two Killing horizons in the region of positive $X$. Moreover, this model exhibits constant dilaton vacua, i.e., solutions where $X = \text{constant}$, for $X = 0$ and for $X = \pm \sqrt{Y}$. For positive $Y$, the solution attached to $X = 0$ is $dS^2$ with $R = -Y$, while the ones attached to $X = \pm \sqrt{Y}$ are $AdS^2$ with $R = 2Y$. The latter coincide with the solutions discussed
in Sec. 4.1 (because there \( X \) has been rescaled to 1, which means \( Y = 1 \) and thus reproduces (32) with \( \lambda = 1 \)). Thus, also the constant dilaton vacua are covered correctly by the action (49b). From a two-dimensional point of view there has to be a kink-like solution interpolating between these constant dilaton vacua, by full analogy to Ref. 1. However, the vacuum \( X = 0 \), although regular in two dimensions, is singular in three dimensions as the line element (44a) degenerates for \( X = 0 \). This explains the absence of kink-like solutions for the present case.

For sake of completeness we mention that by using the potentials

\[
\tilde{U}(\tilde{X}) = \frac{2}{\tilde{X}}, \quad \tilde{V}(\tilde{X}) = -\frac{1}{2\tilde{X}} - 4M\tilde{X}^3, \tag{50a}
\]

the alternative action

\[
\tilde{I}_2 = \frac{1}{8\pi^2} \int d^2 x \sqrt{-g} \left( \tilde{X} R + \frac{2}{\tilde{X}} (\nabla \tilde{X})^2 + \frac{1}{2\tilde{X}} + 4M\tilde{X}^3 \right) \tag{50b}
\]

again exchanges the respective roles of the constant of motion and the parameter of the action, leading to the line element

\[
ds^2 = \frac{1}{\tilde{X}^2} \left( 2dud\tilde{X} + \left( \frac{1}{4\tilde{X}^2} - 2M\tilde{X}^2 - \frac{Y}{2} \right) du^2 \right). \tag{50c}
\]

The transformation \( \tilde{X} = -1/X \) brings (50c) back into the form (49a).

### 6. Conclusions

A Kaluza–Klein reduction of the \( n \)-dimensional conformal tensor (the Weyl tensor in \( n \geq 4 \)) led to interesting equations (17a) for conformal flatness of the \( n \)-dimensional space, which may be interpreted as equations of motion of some \( (n-1) \)-dimensional Einstein–Maxwell like theory. For \( n = 4 \) drastic simplifications occurred (22). The second of these equations, (22b), exhibited the existence of a three-dimensional Killing vector \( f^\mu \), the dual field strength [cf. (20)]. Depending on the properties of \( f^\mu \), the existence of further Killing vectors in three dimensions could be shown, all of which can be lifted to four dimensions:

- For \( f^\mu = 0 \), the three-dimensional space is maximally symmetric and thus has six Killing vectors.
- For generic \( f^\mu \neq 0 \) and \( d_\mu f_\nu \) not vanishing identically, the existence of a dual Killing vector \( F^\mu \) could be shown (26b).
- For space-like \( f^\mu \neq 0 \) and \( d_\mu f_\nu = 0 \), we could show that a further reduction to two-dimensions is possible and that the two-dimensional space is maximally symmetric. Therefore, in addition to \( f^\mu \) there are three Killing vectors.
- For time-like \( f^\mu \neq 0 \) and \( d_\mu f_\nu = 0 \), one can repeat the analysis of the previous case and also ends up with three Killing vectors in addition to \( f^\mu \).
- The case of light-like \( f^\mu \neq 0 \) and \( d_\mu f_\nu = 0 \) is not considered here.
So even in the most generic case there are in total three Killing vectors in the four-dimensional theory: the first one is an input in the Kaluza–Klein Ansatz (1), the second one, $f^\mu$, emerges from the equations of motion, and the third one, $F^\mu$, is dual to the second one. This resembles somewhat the Schwarzschild situation while proving the Birkhoff theorem: with only the Killing vectors of spherical symmetry as input the vacuum Einstein equation implies another Killing vector related to staticity.

We proceeded to find all solutions based upon the Ansatz (28) and could reduce the system to two dimensions. All solutions fell into either of three classes depending on the properties of $f^\mu$: if it was covariantly constant we obtained maximally symmetric two-dimensional subspaces (33); for an intrinsically two-dimensional Killing vector we ended up with (39); finally, if $f^\mu$ had no two-dimensional components at all, we were led to (44). The dual Killing vector $F^\mu$ existed for the latter two cases and always was orthogonal to $f^\mu$, $F^\mu f_\mu = 0$. All our four-dimensional line elements were conformally flat by construction, but we have not presented a coordinate transformation which makes this manifest, $ds^2 = e^{2\sigma} \eta_{MN} dx^M dx^N$. This purely technical exercise is not at all trivial.

In the last part of our paper we constructed various two-dimensional dilaton gravity actions, (47b), (48b), (49b), and (50b), which led to equations of motion, the solutions of which reproduced the three classes mentioned above. This was remarkable insofar as there does not seem to be a straightforward way to obtain an action whose equations of motion are given by $C^{KLMN} = 0$. The closest thing to such an action in the generic case is the well-known quadratic one,

$$\int d^n x \sqrt{|g|} C_{KLMN} C^{KLMN},$$

which can be reduced to an $(n-1)$-dimensional action after inserting (16a) and integrating out the $n$th coordinate. The ensuing equations of motion are rather complicated, however, so we restrict ourselves to the simpler case $n = 4$. The reduced action (dropping the overall constant)

$$\int d^3 x \sqrt{g} \left( c_{\mu\nu} c^{\mu\nu} - \frac{1}{4} K_{\mu\nu} K^{\mu\nu} \right)$$

depends on the traceless symmetric tensors $c_{\mu\nu}$ introduced previously in (16b) and

$$K_{\mu\nu} = d_\mu f_\nu + d_\nu f_\mu.$$

A special class of solutions is determined by the equations $c_{\mu\nu} = K_{\mu\nu} = 0$, which are identical to conditions (22). More general solutions are possible, but we have not attempted to construct them. Thus, this quadratic action encompasses all our solutions but also provides additional ones.

Finally, we would like to comment on the relations between various two-dimensional actions presented in this paper. The feature that two different actions describe the same set of classical solutions, but with the role of the constant of motion and the parameter in the action exchanged, has been a recurring theme.
This property readily generalizes to a large class of dilaton gravity models and appears to be a fully fledged duality. It deserves further study.\textsuperscript{6}

Acknowledgments

D. Grumiller would like to thank Alfredo Iorio for enjoyable discussions on gravitational Chern–Simons terms and their Kaluza–Klein reduction.

This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under the cooperative research agreement DEFG02-05ER41360. D. Grumiller has been supported by project GR-3157/1-1 of the German Research Foundation (DFG) and by the Marie Curie Fellowship MC-OIF 021421 of the European Commission under the Sixth EU Framework Programme for Research and Technological Development (FP6).

Appendix A. Curvature in Two and Three Dimensions

Here are details for two-dimensional geometries of the form (35). The metric components read

\[ g_{uu} = e^{Q}K, \quad g_{uX} = e^{Q}, \quad g_{XX} = 0. \]  
(A.1)

The Killing vector components are

\[ f^{u} = 1, \quad f^{X} = 0, \quad f_{u} = e^{Q}K, \quad f_{X} = e^{Q}. \]  
(A.2)

The two-dimensional Ricci scalar is given by

\[ R = V' - 2UV - U'e^{-Q}K, \]  
(A.3)

where \( U, V \) are defined in (46a). Derivatives of \( X \),

\[ \Box X = -V, \quad (\nabla X)^{2} = -e^{-Q}K, \]  
(A.4)

and

\[ \nabla_{u}\nabla_{u}\ln X = K\nabla_{u}\nabla_{X}\ln X = \frac{K}{2X}(KU - K'), \quad \nabla_{X}\nabla_{X}\ln X = \frac{1}{X^2}(XU - 1) \]

are needed to determine the three-dimensional geometric quantities (29) since \( \phi \) is some function of \( X \). In our problem

\[ \phi = X^{\alpha}, \quad \alpha = \frac{1}{2} \text{ or } 1. \]  
(A.5)

Thus, one obtains the three-dimensional Ricci scalar

\[ r = V' + 2V\left(\frac{\alpha}{X} - U\right) + e^{-Q}K\left(\frac{2\alpha(\alpha - 1)}{X^2} - U'\right), \]  
(A.6a)

and the traceless portion of the three-dimensional Ricci tensor (\( \hat{r}_{\mu\nu} := r_{\mu\nu} - \frac{1}{3}g_{\mu\nu}r \))

\[ \hat{r}_{uu} = K\hat{r}_{uX}, \]  
(A.6b)
\[ \hat{r}_{uX} = \frac{1}{6} V' e^Q - \frac{1}{6} V e^Q \left( \frac{\alpha}{X} + 2U \right) + \frac{1}{6} K \left( \frac{4\alpha(1-\alpha)}{X^2} - U' - \frac{3\alpha}{X} U \right), \]  
(A.6c)

\[ \hat{r}_{XX} = \frac{\alpha}{X^2} (1 - \alpha - XU), \]  
(A.6d)

\[ \hat{r}_{u\theta} = \hat{r}_{\theta u} = 0, \]  
(A.6e)

\[ \hat{r}_{\theta\theta} = \frac{X^{2\alpha}}{3} V' - \frac{X^{2\alpha-1}}{3} V (\alpha + 2XU) + \frac{X^{2\alpha-2}}{3} e^{-Q} K (\alpha(1-\alpha) - X^2 U'). \]  
(A.6f)

Appendix B. Tracefree Combination of Field Strengths

For the Killing vector (A.2) with \( f^\theta = 0 \) the tracefree combination of field strengths reads (\( \hat{t}_{\mu\nu} := f_{\mu} f_{\nu} - \frac{1}{3} g_{\mu\nu} f^2 \))

\[ \hat{t}_{uu} = K \hat{t}_{uX} = \frac{2}{3} K^2 e^{2Q}, \quad \hat{t}_{XX} = e^{2Q}, \]  
(B.1a)

\[ \hat{t}_{u\theta} = \hat{t}_{X\theta} = 0, \quad \hat{t}_{\theta\theta} = \frac{1}{3} X^{2\alpha} e^{Q} K, \]  
(B.1b)

where \( \alpha \) is the exponent defined in (A.5). For the Killing vector \( f^\theta = 1, f^\alpha = 0 \), one obtains

\[ \hat{t}_{uu} = K \hat{t}_{uX} = \frac{1}{3} K e^{Q} X^{2\alpha}, \quad \hat{t}_{XX} = 0, \]  
(B.2a)

\[ \hat{t}_{u\theta} = \hat{t}_{X\theta} = 0, \quad \hat{t}_{\theta\theta} = \frac{2}{3} X^{4\alpha}. \]  
(B.2b)

References