Daniel Grumiller May 12^{th} 2021

Gravity and holography in lower dimensions II

(9.1) **Logarithmic branch point in OTOC**
In the discussion of OTOCs we saw that the 4-point correlator $\langle W(t+i\epsilon_1)V(i\epsilon_3)W(t+i\epsilon_2)V(i\epsilon_4)\rangle$ in a CFT$_2$is proportional to a sum involving the hypergeometric function ${}_2F_1(h,h,2h;z)$ where h is the weight of any primary in the theory. Make explicit that there is a logarithmic branch cut starting at the branch point $z=1$ by using hypergeometric identities, assuming the weights h are positive integers.
[Bonus level: show this also when h are positive half integers.]

(9.2) **OTOC discussion in large c CFT$_2$**
Use the results stated in the lectures for general 4-point functions in a thermal CFT$_2$ expressed in terms of a vacuum 4-point function together with the function of the conformal cross ratios, which in the large c limit is given by the Virasoro identity block $f(z,\bar{z})=\mathcal{F}(z)\bar{\mathcal{F}}(\bar{z})$ where $\mathcal{F}(z)\approx(1-\frac{24\pi i h_W}{cz})^{-2h_V}$ and $\bar{\mathcal{F}}(\bar{z})\approx 1$, to show the key result ($t\gg x$)

$$\frac{\langle W(t+i\epsilon_1,x)V(i\epsilon_3,0)W(t+i\epsilon_2,x)V(i\epsilon_4,0)\rangle_\beta}{\langle W(t+i\epsilon_1,x)W(t+i\epsilon_2,x)\rangle_\beta\,\langle V(i\epsilon_2,0)V(i\epsilon_4,0)\rangle_\beta}\approx\Big(1+\frac{24\pi i\,h_W}{\epsilon_{12}^*\epsilon_{34}}e^{\frac{2\pi}{\beta}(t-t_*-x)}\Big)^{-2h_V}$$

where h_W, h_V are the conformal weights of the operators W, V, and $t_*=\frac{\beta}{2\pi}\ln c\gg t$ is the scrambling time $[i\epsilon_{nm}:=\exp\big(\frac{2\pi i\epsilon_m}{\beta}\big)-\exp\big(\frac{2\pi i\epsilon_n}{\beta}\big)]$.

(9.3) **Chaos in JT gravity and holographic Lyapunov exponent**
Holographically calculate the Lyapunov exponent using the JT model as follows. Consider an AdS$_2$ black hole of mass M and assume that you add an infalling massless particle of energy $\delta M\ll M$, mimicking a perturbation on the field theory side. Consider additionally some outgoing signal near the horizon, both in the unperturbed geometry and in the backreacted one. Show that the time-delay δT generated by this backreaction is given by

$$\delta T\propto e^{\lambda_L\,\delta t}$$

where δt is the expected waiting time (without backreaction). Calculate λ_L and show that it saturates the chaos bound $\lambda_L=2\pi/\beta$.

These exercises are due on June 1$^{\text{st}}$ 2021.

Hints/comments:

- This is a pure math exercise. There are numerous ways you can proceed, and mine most likely is not the quickest one. I used complete induction by showing first ${}_2F_1(1,1,2;z) = -\ln(1-z)/z$, making explicit the logarithmic branch cut starting at $z=1$, and then by using relations between contiguous functions (where the indices a, b and /or c are shifted by integers). Thus, up to additive and multiplicative ratios of polynomials also ${}_2F_1(h,h,2h;z)$ for positive integer h contains a term $\ln(1-z)$.
  [For the Bonus level I used the same arguments, with the starting point ${}_2F_1(\frac{1}{2},\frac{1}{2},1;z) = \frac{2}{\pi}\,K(z)$, where $K(z)$ is the complete elliptic integral of the first kind. Then I used that $K(z)$ has a log branch at $z=1$.]

- This is a pure CFT$_2$ exercise. Use the relation between thermal and vacuum correlators $\langle \mathcal{O}(t,x)\dots\rangle_\beta = \langle \mathcal{O}(z,\bar z)\dots\rangle$ where $z = e^{2\pi/\beta(x+t)}$ and $\bar z = e^{2\pi/\beta(x-t)}$ and the general result for 4-point functions

  $$\langle W(z_1,\bar z_1)V(z_3,\bar z_3)W(z_2,\bar z_2)V(z_4,\bar z_4)\rangle = z_{12}^{-2h_W}\,\bar z_{12}^{-2\bar h_W}\,z_{34}^{-2h_V}\,\bar z_{34}^{-2\bar h_V}\;f(z,\bar z)$$

  where $f(z,\bar z)$ depends on the conformal cross-ratios $z = z_{12}z_{34}/(z_{13}z_{24})$ and $\bar z = \bar z_{12}\bar z_{34}/(\bar z_{13}\bar z_{24})$. Use also the standard result for 2-point functions, $\langle W(z_1,\bar z_1)W(z_2,\bar z_2)\rangle = z_{12}^{-2h_W}\,\bar z_{12}^{-2\bar h_W}$ and similarly for V. This should establish the result $f(z,\bar z)$ for the desired ratio of correlators. Finally, use the explicit result for $f(z,\bar z) \approx \mathcal{F}(z)$ given in the exercise and exploit/show that for $t \gg x$ you get $z \approx -\exp\big(\frac{2\pi}{\beta}(x-t)\big)\epsilon_{12}^*\epsilon_{34}$.

- This is a pure 2d dilaton gravity exercise. You need to calculate the lengths of suitable geodesics, exploiting that the dominant contribution comes from near the horizon. The essence of this exercise is captured by the Figure below. B ($\tilde B$) is the intersection of the shockwave with the initial (new) cutoff surface at time t_1 ($\tilde t_1$). C ($\tilde C$) is the intersection of the outgoing signal with the initial (new) cutoff surface at time t_2 ($\tilde t_2$). For the expected waiting time you should get $\delta t = t_2 - t_1 = \frac{\beta}{2\pi}\ln\frac{1}{\varepsilon} + \dots$, and for the time-delay $\delta T = \tilde t_2 - t_2 \propto \delta M/\epsilon + \dots$.

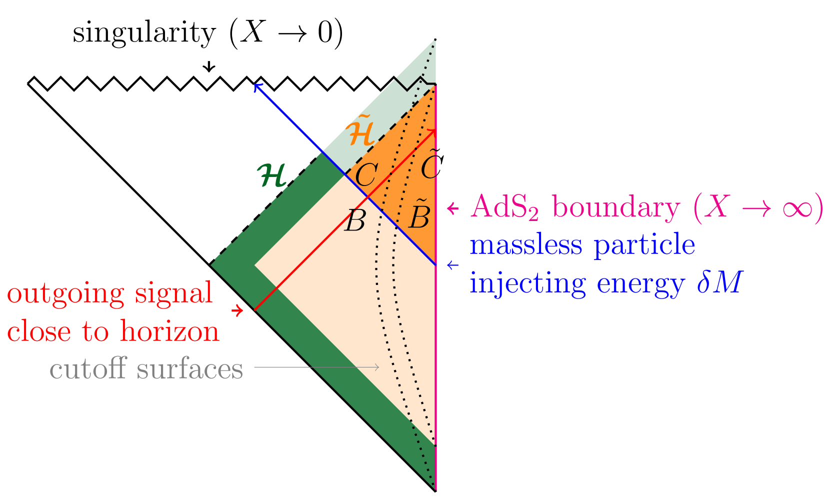

Figure 1: Shockwaves in AdS$_2$ black holes. The horizon $\mathcal{H}$ grows to $\tilde{\mathcal{H}}$ after the shock, increasing the black hole mass from M to $M+\delta M$. Extrapolations of spacetime regions beyond their regime of validity are weakly colored.