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Gravity and holography in lower dimensions I

(9.1) **Infinitesimal Schwarzian derivative in asymptotically AdS$_3$**
Derive how the state-dependent functions transform under asymptotic Killing vectors for asymptotically AdS$_3$ spacetimes subject to Brown–Henneaux boundary conditions.

(9.2) **Mass and angular momentum of BTZ**
From BTZ (for unit AdS radius) in Fefferman–Graham form

$$\mathrm{d}s_{\text{BTZ}} = \mathrm{d}\rho^2 + \left(e^{2\rho}\,\gamma_{ij}^{(0)} + \gamma_{ij}^{(2)} + \dots\right)\,\mathrm{d}x^i\,\mathrm{d}x^j$$

calculate the holographically renormalized Brown–York stress tensor

$$T_{ij}^{\text{BY}} = \frac{1}{8\pi G_N}\left(\gamma_{ij}^{(2)} - \gamma_{ij}^{(0)}\,\mathrm{Tr}\,\gamma^{(2)}\right)$$

for all values of mass and angular momentum parameters. Derive expressions for conserved mass $M = \oint \mathrm{d}\varphi\, T_{tt}^{\text{BY}}$ and angular momentum $J = \oint \mathrm{d}\varphi\, T_{t\varphi}^{\text{BY}}$. Check also their values for global AdS$_3$.

(9.3) **Boundary gravitons on global AdS$_3$**
Consider linearized fluctuations, $g_{\mu\nu} = g_{\mu\nu}^{\text{AdS}} + \psi_{\mu\nu}$, around global AdS,

$$\mathrm{d}s^2_{\text{AdS}} = \mathrm{d}\rho^2 - \cosh^2\rho\;\mathrm{d}t^2 + \sinh^2\rho\;\mathrm{d}\varphi^2 \qquad \varphi \sim \varphi + 2\pi\,.$$

Find all normalizable left-moving linearized fluctuations $\psi_{\mu\nu}$ that obey the SL(2)×SL(2) primary conditions $(L_1^{\pm}h)_{\mu\nu} = 0$ where $L_n^{\pm}$ are the six Killing vectors of global AdS$_3$

$$\begin{aligned}
L_0^{\pm} &= i\partial_{\pm} \\
L_{-1}^{\pm} &= ie^{-ix^{\pm}}\left(\coth(2\rho)\,\partial_{\pm} - \frac{1}{\sinh(2\rho)}\,\partial_{\mp} + \frac{i}{2}\,\partial_{\rho}\right) \\
L_1^{\pm} &= ie^{ix^{\pm}}\left(\coth(2\rho)\,\partial_{\pm} - \frac{1}{\sinh(2\rho)}\,\partial_{\mp} - \frac{i}{2}\,\partial_{\rho}\right)
\end{aligned}$$

with $x^{\pm} = t \pm \varphi$. By the attribute “left-moving” we mean $(L_0^{-}\psi)_{\mu\nu} = 0$ and $(L_0^{+}\psi)_{\mu\nu} = h^{+}\psi_{\mu\nu}$, where the weight has to be positive, $h^{+} > 0$, for the mode to be called “normalizable”.

These exercises are due on January 12th 2021.

Hints:

- The title of this exercise is a strong hint. If you did exercise (7.2) there is little new work you have to do; just check how the asymptotic Killing vectors change the $\mathcal{O}(1)$ terms in the metric. In case you did not do exercise (7.2) you can take the result for the asymptotic Killing vectors provided in the corresponding hint and apply it to metrics obeying Brown–Henneaux boundary conditions. If you are careful with factors you should even be able to read off the Brown–Henneaux central charge $c = 3\ell/(2G_N)$ from your result.

- Read off the Fefferman–Graham expansion matrices $\gamma_{ij}^{(0)}$ and $\gamma_{ij}^{(2)}$ from the BTZ metric given in the lectures, after redefining your radial coordinate suitably. Insert them into the result for the Brown–York stress tensor. Regarding global AdS$_3$, recall that its metric reads

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \mathrm{d}\rho^2 - \cosh^2\rho \, \mathrm{d}t^2 + \sinh^2\rho \, \mathrm{d}\varphi^2 \qquad \varphi \sim \varphi + 2\pi \, .$$

- Work in a gauge where $\psi_{\mu-} = 0$ and exploit that ψ solving the linearized Einstein equations implies $(C_2^+ + C_2^- + 2)\psi = 0$, where $C_2^\pm = \frac{1}{2}\,(L_1^\pm L_{-1}^\pm + L_{-1}^\pm L_1^\pm) - (L_0^\pm)^2$ is the quadratic Casimir. Applying the ancient wisdom of Fourier transforming when you do not know what else to do you can start with the separation ansatz

$$\psi_{\mu\nu}(h^+, h^-) = e^{-ih^+x^+ - ih^-x^-} \begin{pmatrix} F_{++}(\rho) & 0 & F_{+\rho}(\rho) \\ 0 & 0 & 0 \\ F_{+\rho}(\rho) & 0 & F_{\rho\rho}(\rho) \end{pmatrix}_{\mu\nu}$$

so that you work with $L_0^\pm$ eigenmodes, $L_0^\pm\psi = h^\pm\psi$, and have implemented already the required gauge conditions $\psi_{\mu-} = 0$. The left-moving condition sets one of the weights to zero, $h^- = 0$. The Einstein equations (using the quadratic Casimir) fix the (by normalizability positive!) other weight, $h^+ = 2$. The remaining steps are to solve the Killing equations corresponding to the two primary conditions, using the ansatz above. Note that some of the equations linearly combine to algebraic relations between the three functions $F_{\mu\nu}(\rho)$. One of the $++$ component equations allows to immediately determine $F_{++} \propto \tanh^2\rho$ by simple integration. In the end this procedure yields a unique result for ψ, up to an overall factor.
Note: the attribute "boundary gravitons" is justified since we know that in the bulk there are no physical degrees of freedom, i.e., no gravitational waves can propagate through the bulk; however, at the boundary some of the pure gauge excitations can become physical.