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Gravity and holography in lower dimensions I

(9.1) **Holographic analysis of quantum Hall system**
Consider abelian Chern–Simons theory, which is the low energy effective action for electrodynamics in 2+1 dimensions, arising e.g. in the description of the quantum Hall effect. Assume the quantum Hall system lives on a disk (with coordinates $\varphi \sim \varphi + 2\pi$ and $\rho \in [0, \rho_b]$), so that the Chern–Simons theory lives on a cylinder and make the ansatz for the abelian connection 1-form

$$A = \mu(t)\ \mathrm{d}t + \mathcal{J}(t,\,\varphi)\ \mathrm{d}\varphi + \mathrm{d}\rho\,.$$

where the chemical potential $\mu(t)$ is fixed, $\delta\mu = 0$, while the response function $\mathcal{J}$ is allowed to vary, $\delta\mathcal{J} \neq 0$. Determine the asymptotic symmetry algebra, including its central extension.

(9.2) **Flat space limit of Virasoro**
Start with the asymptotic symmetry algebra of AdS_3 Einstein gravity with Brown–Henneaux boundary conditions (here written as commutator algebra)

$$[L_n^\pm,\, L_m^\pm] = (n-m)\, L_{n+m}^\pm + \frac{\ell}{8G}\left(n^3 - n\right)\delta_{n+m,\,0} \qquad\qquad [L_n^+,\, L_m^-] = 0$$

and make a change of basis

$$L_n := L_n^+ - L_{-n}^- \qquad\qquad M_n := \frac{1}{\ell}\left(L_n^+ + L_{-n}^-\right).$$

The generators L_n, M_n still generate two copies of Virasoro for finite AdS radius ℓ, albeit in an unusual basis. Take the flat space limit $\ell \to \infty$, which leads to an İnönü–Wigner contraction of the algebra, and write down the commutation relations after you have taken this limit. The resulting algebra is the flat space limit of Virasoro.

(9.3) **Virasoro vacuum and descendants**
Take again the asymptotic symmetry algebra of AdS_3 Einstein gravity with Brown–Henneaux boundary conditions as commutator algebra [see exercise (9.2) above] and define the Virasoro vacuum as a state $|0\rangle$ obeying

$$L_n^\pm |0\rangle = 0 \qquad \forall n \geq -1\,.$$

Consider specific descendants $|n^+, n^-\rangle$ of the vacuum defined by

$$|n^+, n^-\rangle := L_{-n^+}^+ L_{-n^-}^- |0\rangle \qquad n^\pm \in \mathbb{N}$$

and calculate their energy eigenvalues (the Virasoro Hamiltonian is given by $H = L_0^+ + L_0^-$).

These exercises are due on December 4^{th} 2018.

Hints:

- Start by deriving consequences from the equations of motion, $\mathrm{d}A = 0$. Then derive the most general boundary condition preserving transformation $\delta_\epsilon A = \delta\mathcal{J}\,\mathrm{d}\varphi$. Finally, insert your results into the general expression for the variation of canonical boundary charges in Chern–Simons theories, equation (61) in chapter 6. At this point you should take a look at exercise (8.2).

  Note: The resulting algebra is also known as the loop algebra of the underlying gauge algebra (in this case $u(1)$). It is a general feature of Chern–Simons theories that for relaxed boundary conditions where all generators contain functions that are allowed to fluctuate the resulting asymptotic symmetry algebra is the loop algebra of the underlying gauge algebra. See `1608.01308` *for an application to AdS_3 Einstein gravity.*

- Keep initially all terms containing the AdS radius ℓ when writing the algebra entirely in terms of the new generators L_n and M_n. Then take the limit $\ell \to \infty$.

  Note: The resulting algebra is known as (centrally extended) BMS_3 algebra, which is the asymptotic symmetry algebra of flat space Einstein gravity in three spacetime dimensions. See `gr-qc/0610130`.

- This exercise is short, and so is this hint. Act with H on the given state, and use the Virasoro algebra as well as $L_0^\pm|0\rangle = 0$.

  Note: On the CFT_2 side these states are naturally interpreted as descendants of the vacuum. As the exercise shows they have positive energy. On the AdS_3 side the analogue of these states can be interpreted as "boundary graviton" excitations that are generated when acting with suitable combinations of asymptotic Killing vectors on the global AdS_3 vacuum.