Gravity and holography in lower dimensions I

(8.1) **Curvature a la Cartan**
Take the metric

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}r - \frac{r^2 - r_+^2}{\ell^2}\,\mathrm{d}u^2 + r^2\,\mathrm{d}\varphi^2$$

construct a dreibein, solve for the spin-connection by imposing vanishing torsion and calculate the curvature 2-form. Which geometry is this locally? (and in case you know, which geometry is this globally?)

(8.2) **Orbifolds of AdS$_3$**
Take AdS$_3$ (with AdS-radius ℓ) in embedding coordinates

$$\mathrm{d}s_4^2 = -\,\mathrm{d}u^2 - \mathrm{d}v^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \qquad -u^2 - v^2 + x^2 + y^2 = -\ell^2$$

and consider identifications of AdS$_3$ (a.k.a. orbifolds) by

$$e^{2\pi\xi}$$

with the Killing vector ($r_+ > 0$)

$$\xi = \frac{r_+}{\ell}\,(x\partial_u + u\partial_x)\,.$$

Derive the conditions for the identification to be free from closed time-like curves.

How does the Killing vector ξ and the metric look like in the coordinates t, r, φ defined by

$$u = \ell\,\frac{r}{r_+}\,\cosh\Big(\frac{r_+}{\ell}\,\varphi\Big) \qquad\qquad x = \ell\,\frac{r}{r_+}\,\sinh\Big(\frac{r_+}{\ell}\,\varphi\Big)$$
$$y = \ell\,\frac{\sqrt{r^2 - r_+^2}}{r_+}\,\cosh\Big(\frac{r_+}{\ell^2}\,t\Big) \qquad v = \ell\,\frac{\sqrt{r^2 - r_+^2}}{r_+}\,\sinh\Big(\frac{r_+}{\ell^2}\,t\Big)$$

for $r > r_+ > 0$? BONUS QUESTION: What does the identification above imply for the angular coordinate φ?

(8.3) **BTZ black hole ergo region**
Do BTZ black holes, given by the three-dimensional metric ($r_+ \geq r_- \geq 0$; $r^{\pm}, \ell \in \mathbb{R}^+$; $t, r \in \mathbb{R}$; $\varphi \sim \varphi + 2\pi$)

$$\mathrm{d}s^2_{\text{BTZ}} = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2}\,\mathrm{d}t^2 + \frac{\ell^2 r^2\,\mathrm{d}r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2\,\Big(\mathrm{d}\varphi - \frac{r_+ r_-}{\ell r^2}\,\mathrm{d}t\Big)^2$$

have an ergo region? Are you sure?

These exercises are due on December 15th 2020.

Hints:

- You could answer both questions in the end almost without doing any calculations. However, for calculating the curvature 2-form you need to do some calculations, so it is best to start with dreibein, then solve torsion equals zero for the spin-connection and then insert into the definition of the curvature 2-form. If you know what you must get in advance you have a nice cross-check on possible mistakes. Part of the calculation is analogous to the two-dimensional example in the lectures, so you could use similar techniques, but in one dimension higher. Specifically, I suggest you use the ansatz $e^+ = \mathrm{d}u$, $e^- = -\,\mathrm{d}r - \frac{1}{2}\,K(r)\,\mathrm{d}u^2$ similar to the lectures (with a suitable choice for the function $K(r)$), together with an obvious ansatz for the third dreibein, $e^3 = r\,\mathrm{d}\varphi$, with the Minkowski metric $\eta_{\pm\mp} = 1 = \eta_{33}$ (all other components of η_{ab} vanish). As a final hint: in case you did all the calculations but still have difficulties in answering the questions recall that a result like $R^{ab} \propto e^a \wedge e^b$ implies that you have a constant curvature spacetime.

- Closed timelike curves exist if and only if the Killing vector ξ is timelike. The first part of this exercise is rather short, but you are encouraged to go further by reading section 3.2 in `gr-qc/9302012`. The second part is straightforward — just apply the coordinate transformation to calculate the Killing vector and the metric. If you have the result for the Killing vector the bonus question can be answered without any further calculation.

- There are two ways you could define an ergo region, hence the follow-up question of how sure you are: 1. with respect to the asymptotic time variable t, checking for zero of the norm of the Killing vector ∂_t; 2. with respect to an arbitrary Killing vector $\partial_t + A\partial_\varphi$ (where A is some constant chosen such that this Killing vector remains timelike everyhwere outside the event horizon). The first case is straightforward and leads to a short answer to the first question (never mind answering the second question). The second case is more interesting and leads to an even shorter answer (by one letter).