Daniel Grumiller November 20^{th} 2018

Gravity and holography in lower dimensions I

(8.1) **Canonical analysis of mechanics model**
Consider the Lagrangian ($\epsilon_{ij} = -\epsilon_{ji}$, $\epsilon_{12} = 1$)

$$L(q_i,\, q_3,\, \dot{q}_i,\, \dot{q}_3) = \epsilon_{ij}\, q_i \dot{q}_j - V(q_i) \qquad i, j \in \{1, 2\}$$

and derive the canonical Hamiltonian, the total Hamiltonian, all constraints (which of them are first and which second class?) as well as the dimension of the physical phase space.

(8.2) **Canonical realization of simple set of asymptotic symmetries**
Suppose you have a physical system in three spacetime dimensions whose canonical boundary charges vary according to ($\varphi \sim \varphi + 2\pi$)

$$\delta_{\epsilon_1} Q[\epsilon_2] = \frac{k}{2\pi} \oint_{S^1} \mathrm{d}\varphi\, \epsilon_2(\varphi)\, \delta_{\epsilon_1} \mathcal{J}(\varphi) \qquad\qquad \delta_{\epsilon_1} \mathcal{J} = \frac{\mathrm{d}\epsilon_1}{\mathrm{d}\varphi}$$

where k is a coupling constant, ϵ is the (state-independent) transformation parameter, and $\mathcal{J}$ a state-dependent function. Introduce Fourier modes and determine the canonical realization of the asymptotic symmetries implied by the equation above, written as an algebra for the corresponding Fourier modes. Determine in particular whether this algebra has a central charge and if so, how it looks like.

(8.3) **Central extension of Galilei algebra**
Consider the Galilei algebra in two spatial (and one time) dimensions, generated by time-translations H, spatial translations P_i, spatial rotations J and Galilean boosts K_i, whose non-zero commutators read

$$[J,\, P_i] = \epsilon_{ij}\, P_j \qquad [J,\, K_i] = \epsilon_{ij}\, K_j \qquad [K_i,\, H] = P_i\,.$$

Determine a non-trivial central extension of this algebra.

These exercises are due on November 27^{th} 2018.

Hints:

- Follow the canonical algorithm summarized in the appendix to chapter 6 of the lecture notes. Note that the unphysical phase space is 6-dimensional, spanned by the three canonical coordinates q_i, q_3 and the three corresponding canonical momenta.

- Start with the key equation (30) in chapter 6 of the lecture notes, which relates the gauge variation of the boundary charges to their Poisson brackets. Defining Fourier modes of the charges e.g. as $J_n := Q[e^{in\varphi}]$ and using $\oint \mathrm{d}\varphi \, e^{i(n+m)\varphi} = 2\pi \, \delta_{n+m,\,0}$ this exercise should take only a handful of lines.

- You can either proceed by trial and error (the algebra has several non-trivial central extensions, so chances are that you get lucky), which can be a quick way to solve this exercise, or by finding all central extensions and eliminating trivial ones by a change of basis. In the latter case write down all commutators and add central extensions where possible, using δ_{ij} and ϵ_{ij} where necessary. Note that commutators that have exactly one free index (like $[H, \, P_i]$) cannot possibly have a central extension, since there is not quantity with a vector index except for the generators. Thus, the most general ansatz for a central extension of the 2+1 dimensional Galilei algebra should lead you to five candidates for non-trivial central extensions, four of which reside in the P_i, K_i-sector. By taking arbitrary linear combinations of translations and boosts you can eliminate three of them (though you may want to keep two, since there are two inequivalent cases of what could be zero); it is conventional to define in the new basis the translation generators such that $[P_i, \, P_j]$ remains zero. Having said all this, recall that the goal of the exercise is to find *a* non-trivial central extension and not all of them. So the exercise can be much shorter than the text in this hint indicates.