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# Gravity and holography in lower dimensions I

(7.1) **Holographic entanglement entropy of global AdS$_3$**
Determine entanglement entropy for an angular interval of length $L$ for the state dual to global AdS$_3$, given by a Bañados geometry with $\mathcal{L}^\pm = -1$.

(7.2) **Asymptotic Killing vectors in asymptotically AdS$_3$**
Derive the asymptotic Killing vectors as well as their Lie-bracket algebra for asymptotically AdS$_3$ boundary conditions

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \Big(e^{2\rho/\ell}\gamma^{(0)}_{\mu\nu} + \gamma^{(2)}_{\mu\nu} + \mathcal{O}\big(e^{-2\rho/\ell}\big)\Big)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

with $\gamma^{(0)}_{\mu\nu} = \eta_{\mu\nu}$ and $\delta\gamma^{(2)}_{\mu\nu} \neq 0$. (Use lightcone gauge $\eta_{+-} = 1$, $\eta_{\pm\pm} = 0$.)

(7.3) **Transformation of non-abelian curvature 2-form**
Take the non-abelian gauge curvature 2-form

$$F = \mathrm{d}A + A \wedge A$$

where $A$ is some non-abelian gauge connection 1-form, $A = T_a A^a_\mu\,\mathrm{d}x^\mu$, with $[T_a, T_b] = f_{ab}{}^c T_c$ generating some non-abelian Lie algebra with structure constants $f_{ab}{}^c$. Derive how $F$ transforms under non-abelian gauge transformations

$$A \to \hat{A} = g^{-1}\,\mathrm{d}g + g^{-1}Ag$$

generated by some group element $g$. If $F = 0$ what does this imply for the transformed $\hat{F}$? Derive also the infinitesimal version of the gauge transformation of the connection $A$ by assuming $g$ is close to the identity element.

**These exercises are due on December 8th 2020.**

Hints:

- There are two ways to solve this. Either determine the lengths of suitable geodesics in global $AdS_3$, analogous to exercise (6.3), or use the uniformized result for entanglement entropy stated in the lectures in terms of the bilinears $\ell^\pm = \psi_1^\pm(x_1^\pm)\psi_2^\pm(x_2^\pm) - \psi_2^\pm(x_1^\pm)\psi_1^\pm(x_2^\pm)$. To obtain the functions $\psi_{1,2}^\pm$ solve Hill's equation $\psi^{\pm\prime} - \mathcal{L}^\pm\,\psi^\pm = 0$ with $\mathcal{L}^\pm = -1$ and enforce the Wronskian normalization $\psi_2^\pm\psi_1^{\pm\prime} - \psi_1^\pm\psi_2^{\pm\prime} = \pm 1$. Use $x_1^+ - x_2^+ = -x_1^- + x_2^- = L$, after exploiting simple trigonometric identities like $\sin a\cos b - \cos a \sin b = \sin(a-b)$. In either of the approaches a cross-check of your final result is that for small $L$ you recover the Poincaré patch $AdS_3$ result $S_{\text{EE}} = \frac{c}{3}\,\ln\frac{L}{\varepsilon_{\text{UV}}}$.

- Solve the asymptotic Killing equations, starting with the components that are fixed ($g_{\rho\rho}$) or vanish ($g_{\rho\mu}$) to determine various constraints on the asymptotic Killing vectors (keeping only the leading order at large $\rho$). You should find the result announced in the lectures

  $$\xi = \varepsilon^+(x^+)\partial_+ + \varepsilon^-(x^-)\partial_- - \frac{\ell}{2}\left(\partial_+\varepsilon^+(x^+) + \partial_-\varepsilon^-(x^-)\right)\partial_\rho + \mathcal{O}\big(e^{-2\rho/\ell}\big)$$

  Their Lie-bracket algebra follows straightforwardly. In case you want to introduce Fourier modes you should find two Witt algebras.

- The calculation is straightforward and should fit into less than a handful of lines. Do not forget to answer the questions. For the last one expand $g = e^\varepsilon$ for small $\varepsilon = T_a\varepsilon^a$ up to linear order in $\varepsilon$ and simplify the expression for $\hat{A}$ until you have one term also present in the abelian case and one term that you can express as commutator.