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Gravity and holography in lower dimensions II

(6.1) **Poisson-σ model gauge invariance**
Show that the Poisson-σ model bulk action

$$I_{\text{PSM}}[A_I,\, X^I] = \frac{k}{2\pi} \int \big(X^I \,\mathrm{d}A_I + \tfrac{1}{2}\, P^{IJ}\, A_I \wedge A_J\big)$$

is invariant under the non-linear gauge symmetries

$$\delta_\lambda X^I = P^{IJ}\lambda_J \qquad\qquad \delta_\lambda A_I = -\,\mathrm{d}\lambda_I - \big(\partial_I P^{JK}\big)\, A_J\lambda_K$$

provided the non-linear Jacobi-id's hold, $\sum_{\text{cycl}(I,J,K)} P^{IL}\partial_L P^{JK} = 0$.

(6.2) **First and second order formulations of 2D dilaton gravity**
Given a Poisson-σ model [see the action given in exercise (6.1)] with field content $A_I = (\omega, e_a)$, $X^I = (X, X^a)$ and Poisson tensor

$$P^{aX} = \epsilon^a{}_b X^b \qquad\qquad P^{ab} = -\epsilon^{ab}\,\big(V(X) + \frac{1}{2}\, X^c X_c U(X)\big)$$

show that this first-order model is equivalent to the second-order 2D dilaton gravity action

$$I^{(2)}[g_{\mu\nu,}\, X] \sim \int \mathrm{d}^2x\sqrt{|g|}\,\big(XR - U(X)(\partial X)^2 - 2V(X)\big)$$

where ω is the dualized spin-connection and e_a the zweibein 1-form.

(6.3) **Equivalence of gauge and second order formulations?**
Consider in which sense gauge-theoretic and second order formulations of gravity are equivalent/inequivalent (your considerations may either be phrased in the context of Poisson-σ model vs. 2D dilaton gravity and/or in the context of Chern–Simons vs. 3D Einstein gravity). If you wish to claim that these formulations are inequivalent then construct an example that shows their inequivalence. If you wish to claim their equivalence then find arguments supporting this claim.

These exercises are due on May 11th 2021.

Hints/comments:

- Vary the action, partially integrating if necessary, and insert the gauge variation formulas given in the exercise. The trick is that there is no trick.

- You could start with the simpler case $U(X) = 0$. Integrating out the auxiliary fields X^a then establishes the condition of vanishing torsion, so when you integrate out the spin connection you will find that it must be torsion-free, which converts the curvature 2-form essentially into the Ricci scalar times the volume 2-form (you can be cavalier about signs and factors, if you want — at least, I will not check them in this exercise). Once the $U(X) = 0$ case is understood you can relax this assumption and consider the generic case. Now the torsion 2-form is no longer zero, but given by something proportional to the volume 2-form and the function $U(X)$. You can split the spin connection into a Levi–Civita part and a torsion part, $\omega = e_a * \mathrm{d}e^a - e_a * T^a$ and again replace the term $X\, \mathrm{d}\tilde{\omega}$ essentially by Ricci scalar times volume 2-form, but now you will have extra terms. To eliminate X^a by means of its equation of motion you should then find $\mathrm{d}X \wedge e^c + X^c \epsilon = 0$ where ϵ is the volume 2-form; so basically, X^a are the directional derivatives of the dilaton. Finally, from all this you should conclude that $X^a X_a \sim (\partial X)^2$. If you need more hints consult section 2.2 in `hep-th/0204253`.

- Here are some random comments, which may either be hints or distractions:

  - gauge symmetries generated by λ^I and diffeos generated by ξ^μ are related by $\lambda^I = A^I_\mu \xi^\mu$
  - gauge field configurations $A = 0$ solve the field equations $F = \mathrm{d}A + A \wedge A = 0$
  - for each set of boundary conditions in the gravity formulation there is a corresponding set of boundary conditions in the gauge theoretic formulation
  - in the path integral formulation it may be ok to allow off-shell configurations where the metric degenerates
  - for classical equivalence of different formulations of a theory it is sufficient to show that the physical phase spaces coincide with each other