Daniel Grumiller November $16^{\text{th}}$ 2020

# Gravity and holography in lower dimensions I

(6.1) **Casini–Huerta spacetime diagram**
Prove the formula (20) in the notes on EE, $AD = BC$, that we used in the derivation of the Casini–Huerta $c$-function.

(6.2) **Entanglement entropy of thermal states**
Use the exponential map $z = \exp\big(\frac{2\pi}{\beta}\,w\big)$ to a cylinder of circumference $\beta = T^{-1}$ to derive the thermal result for EE

$$S_A(L;\,T) = \frac{c}{3}\,\ln\left(\frac{\beta}{\pi\varepsilon_{\text{UV}}}\,\sinh\frac{\pi L}{\beta}\right) + \text{const.}$$

from the $T = 0$ result $S_A(L) = \frac{c}{3}\,\ln\frac{L}{\varepsilon_{\text{UV}}} + \text{const.}$ Verify that in the large $T$ limit you get a volume law for EE, $S_A(L;\,T) \propto LT$.

(6.3) **Geodesics in Poincaré patch AdS$_3$**
Just for fun and with no hidden agenda, calculate the geodesic length of an equal time-geodesic

$$S_A(L) = \frac{1}{4G}\int_L \mathrm{d}s$$

for a spatial interval of length $L$ anchored at the asymptotic boundary $z \to 0$ of Poincaré patch AdS$_3$

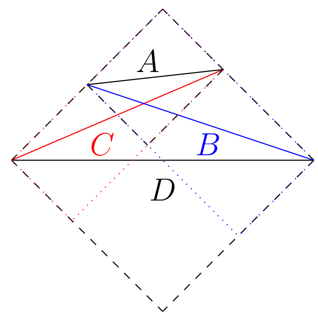
$$\mathrm{d}s^2 = \frac{\ell^2}{z^2}\big(-\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}z^2\big)$$

Since the result will diverge, introduce a small cutoff $z = \varepsilon$ instead of calculating at $z = 0$. Express your result as function of the interval length $L$, the cutoff $\varepsilon$, the AdS radius $\ell$ and Newton's constant $G$.

**These exercises are due on December 1st 2020.**

Hints:

- Either use explicitly coordinates or prove this in a coordinate independent way. Here is again the figure.




- Recall how conformal primaries transform and look up the conformal weights $\Delta_n = \bar{\Delta}_n$ of the twist operators $\Phi_{\pm n}$ in the lecture notes. Work first at the level of the $n^{\text{th}}$ Rényi entropy and then take the limit $n \to 1^+$, like in the lectures. The key formula you need to use is

$$S_A = -\frac{\mathrm{d}}{\mathrm{d}n} \operatorname{tr} \rho_A^n \Big|_{n\to 1^+} = -\frac{\mathrm{d}}{\mathrm{d}n} \Big( \langle \Phi_n(w_1,\, \bar{w}_1) \Phi_{-n}(w_2,\, \bar{w}_2) \rangle \Big)^n \Big|_{n\to 1^+}$$

The UV cutoff can be introduced at the final step on dimensional grounds (why?). You can assume $w_1 - w_2 = \bar{w}_1 - \bar{w}_2 = L$.

- You can either calculate the Christoffels and brute-force solve the geodesic equations with suitable boundary conditions, or you directly use the action functional (convince yourself why this expression is correct!)

$$S_A = \frac{1}{4G}\, 2 \int\limits_{L/2-\mathcal{O}(\varepsilon)}^{0} \mathrm{d}x\, \ell\, \mathcal{L}(z,\, \dot{z})$$

where you should find $\mathcal{L}(z,\, \dot{z}) = \sqrt{1+\dot{z}^2}/z$, with dot denoting $x$-derivatives; then exploit the Noether charge associated with invariance under $x$-translations, $Q = \mathcal{L} - \dot{z}\partial\mathcal{L}/\partial\dot{z}$ and relate it to the maximal $z$ value that can be taken on a geodesic. Finally, note that the interval length is simply given by

$$L/2 - \mathcal{O}(\varepsilon) = \int\limits_{0}^{L/2-\mathcal{O}(\varepsilon)} \mathrm{d}x = \int\limits_{z_{\max}}^{\varepsilon} \frac{\mathrm{d}z}{\dot{z}}\, .$$

And of course there is a hiden agenda! Once you have the final result you'll see...