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Gravity and holography in lower dimensions I

(6.1) **BTZ as orbifold of AdS_3**

Take AdS_3 (with AdS-radius ℓ) in embedding coordinates

$$\mathrm{d}s_4^2 = -\,\mathrm{d}u^2 - \mathrm{d}v^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \qquad -u^2 - v^2 + x^2 + y^2 = -\ell^2$$

and consider identifications of AdS_3 by

$$e^{2\pi\xi}$$

with the Killing vector ($r_+ > 0$)

$$\xi = \frac{r_+}{\ell}\,(x\partial_u + u\partial_x)\,.$$

Derive the conditions for the identification to be free from closed time-like curves.

How does the Killing vector ξ and the metric look like in the coordinates t, r, φ defined by

$$u = \ell\,\frac{r}{r_+}\,\cosh\Big(\frac{r_+}{\ell}\,\varphi\Big) \qquad x = \ell\,\frac{r}{r_+}\,\sinh\Big(\frac{r_+}{\ell}\,\varphi\Big)$$

$$y = \ell\,\frac{\sqrt{r^2 - r_+^2}}{r_+}\,\cosh\Big(\frac{r_+}{\ell^2}\,t\Big) \qquad v = \ell\,\frac{\sqrt{r^2 - r_+^2}}{r_+}\,\sinh\Big(\frac{r_+}{\ell^2}\,t\Big)$$

for $r > r_+ > 0$? BONUS QUESTION: What does the identification above imply for the angular coordinate φ?

(6.2) **Witten black hole**

Consider two-dimensional dilaton gravity with kinetic potential $U = -1/X$ and potential $V \propto X$. Perform a dilaton-dependent Weyl rescaling of the metric (or correspondingly of the first order variables) to a model where $\tilde{U} = 0$ — how does the transformed potential $\tilde{V}$ look like in this frame? Finally, try to obtain the Witten black hole as a special case of spherically symmetric General Relativity in D dimensions — what do you need to choose for the dimension D?

(6.3) **Reissner–Nordström in any dimensions**

Consider Einstein–Maxwell theory in any dimension $D > 3$, assume strict spherical symmetry of metric and gauge field, and integrate out the angular coordinates from the action. How does the effective two-dimensional dilaton-gravity-Maxwell theory look like?

These exercises are due on November 13$^{\text{th}}$ 2018.

Hints:

- Closed timelike curves exist if and only if the Killing vector ξ is spacelike. The first part of this exercise is rather short, but you are encouraged to go further by reading section 3.2 in `gr-qc/9302012`. The second part is straightforward — just apply the coordinate transformation to calculate the Killing vector and the metric. If you have the result for the Killing vector the bonus question can be answered without any further calculation.

- The bulk action is defined in Eq. (17) in section 5 of the lecture notes. Since we probably did not yet discuss the action of Weyl rescalings on the two-dimensional dilaton gravity action there is a small calculation that you have to make to address the first point of the exercise. You will need to use that under dilaton-dependent Weyl rescalings

  $$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\Omega(X)}\, g_{\mu\nu}$$

  the Ricci scalar transforms as

  $$R \to \tilde{R} = e^{-2\Omega(X)} \left(R - 2\nabla^2\Omega(X) \right).$$

  Concerning the last question, just compare the dilaton potentials U, V appearing in the Witten black hole with the ones appearing in spherically symmetric General Relativity in D dimensions (see section 5.2.4 in the lecture notes).

- The gravity part is presented in the lecture notes in section 5.2.4, so all you have to do is to spherically reduce the D-dimensional Maxwell action

  $$I_{\text{Max}} = -\frac{1}{4} \int \mathrm{d}^D x \sqrt{-g^{(D)}}\, F^{\mu\nu} F_{\mu\nu}$$

  where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the $u(1)$ connection A_μ is assumed to be strictly spherically symmetric, meaning that it has no dependence on the angular coordinates nor components in angular directions.