Daniel Grumiller November 9^{th} 2020

Gravity and holography in lower dimensions I

(5.1) **Schwarzian**

Prove that the Schwarzian derivative

$$S(f(z),\,z)=\frac{1}{(\partial f)^2}\Big((\partial f)(\partial^3 f)-\frac{3}{2}\,(\partial^2 f)^2\Big)$$

is annihilated if and only if $f(z)$ generates an SL(2, $\mathbb{R}$) transformation, $f(z)=(az+b)/(cz+d)$. Moreover, show that for infinitesimal transformations $f(z)=z+\varepsilon(z)+\mathcal{O}(\varepsilon^2)$ the Schwarzian derivative expands as

$$S(f(z),\,z)=\partial^3\varepsilon+\mathcal{O}(\varepsilon^2)\,.$$

(5.2) **Virasoro descendants of the vacuum**

As discussed, the vacuum state $|0\rangle$ obeys the highest weight conditions $L_n|0\rangle=0$, $\forall n\geq-1$, where L_n are the Virasoro generators. Consider a generic descendant of the vacuum

$$|n_1,\,n_2,\,\dots,\,n_m\rangle:=L_{-n_1}L_{-n_2}\dots L_{-n_m}|0\rangle\qquad n_i\geq 2\;\forall i=1\dots m$$

and calculate its L_0 eigenvalue, called the 'level' of the descendant. Verify that the number of descendants at level N coincides with the N^{th} Taylor expansion coefficient around $q=0$ of the generating function

$$\prod_{n=2}^{\infty}\frac{1}{1-q^n}=1+q^2+q^3+2q^4+2q^5+4q^6+4q^7+7q^8+\mathcal{O}(q^9)\,.$$

Either prove this generally or verify it explicitly for $N=0,1,\dots,8$.

(5.3) **Cardy formula and asymptotic density of states**

Defining as usual the microcanonical density of states at energy E as $\rho(E)=e^{S(E)}$ show that the asymptotic density of states (large energy and hence large entropy) is correctly reproduced by the Cardy formula, using the standard Laplace transformation between canonical partition function and microcanonical density of states

$$Z[\beta]=\int\limits_0^\infty \mathrm{d}E\,\rho(E)\,e^{-\beta E}\,.$$

These exercises are due on November 24$^{\text{th}}$ 2020.

Hints:

- For the first part one direction is trivial to show: just insert $f(z) = (az+b)/(cz+d)$ into the Schwarzian derivative and show that it vanishes. To verify the other direction you need to solve the differential equation $S(f(z), z) = 0$ for $f(z)$. If you do this with Mathematica you will get the solution immediately; if you do this by hand notice that you can first substitute $g(z) = f'(z)$ to get a non-linear second order ODE for $g(z)$ and then use the substitution $y(z) = 1/\sqrt{g(z)}$ to simplify it to a linear second order ODE, which is solved easily (note that you can always drop overall factors of $g(z)$). The last part of the exercise is straightforward and short.

- Insert into the eigenvalue equation

$$L_0|n_1,\, n_2,\, \ldots,\, n_n\rangle = N|n_1,\, n_2,\, \ldots,\, n_n\rangle$$

  the definition of the vacuum descendant and commute L_0 step-by-step through the Virasoro generators L_{-n_i} using the Virasoro algebra. For the second task you can start (and, if you want, finish) with the explicit verification until level 8, which is a simple and manageable counting problem; e.g. at level four you have the two descendants $L_{-4}|0\rangle$ and $L_{-2}^2|0\rangle$. Proving the correctness of the generating function for any level N is not required for this exercise.

  In case you are ambitious: Euler's generating function for the number of partitions $p(N)$ of the integer N is given by $\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n) q^n$. Proceed from there, noting that there are no L_{-1} descendants of the vacuum. You should be able to conclude that it is then sufficient to prove the following lemma: the number of partitions of N with no parts equal to 1 is $p(N) - p(N-1)$. For more info see section 2 of
  http://www.math.upenn.edu/∼wilf/PIMS/PIMSLectures.pdf.

- Assume that entropy to a good approximation is given by the Cardy formula

$$S(E) \approx 2\pi \sqrt{\frac{cE}{3}}$$

  and use the saddle point approximation to the integral

$$\int \mathrm{d}E\, e^{f(E)} \approx e^{f(E_s)}$$

  where the saddle point value E_s is determined by $f'(E_s) = 0$. Verify that this saddle-point value¸ is consistent with the first law of thermodynamics, $\mathrm{d}E = T\,\mathrm{d}S$. Finally, show that the result above correctly reproduces the high temperature partition function $Z[\beta] \approx \exp(\frac{\pi^2 c}{3\beta})$.