

Black Holes I — Exercise sheet 5

(5.1) Eddington-Finkelstein gauge

Take the Schwarzschild line-element in Schwarzschild gauge, viz.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

and perform a coordinate transformation $v = t + f(r)$ with some suitable function f (which you have to determine) so that you obtain the Schwarzschild line-element in Eddington-Finkelstein gauge, viz.

$$ds^2 = -2 dr dv - \left(1 - \frac{2M}{r}\right) dv^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Is this line-element singular at $r = 2M$?

(5.2) Coordinate transformation of Christoffels

Consider a general coordinate transformation $x' = x'(x)$ and calculate the transformation of the Christoffel symbols (of the first kind). Do they transform as a tensor?

(5.3) Covariant derivative of a constant vector field

Given a smooth manifold in $n \geq 2$ dimensions equipped with an arbitrary metric of signature $(-, +, \dots, +)$ consider a vector field v^μ , whose components in a certain coordinate system (t, x_1, \dots, x_{n-1}) are given by $v^t = 1$ and $v^\mu = 0$ otherwise. Thus, in these coordinates the vector field is a constant vector field and takes the simple form $v^\mu = (1, 0, \dots, 0)$. The main task of this exercise is to calculate the covariant derivative of this vector field, $\nabla_\nu v^\mu$. (When) is it true that $\nabla_\nu v^\mu = 0$ for this vector field v^μ ?

These exercises are due on December 3rd 2019.

Hints:

- This is perhaps the most efficient way to proceed: use the Ansatz $v = t + f(r)$, derive the coordinate differential dv and calculate backwards by starting with the line-element in Eddington-Finkelstein gauge and ending with the line-element in Schwarzschild gauge. Concerning the last question: the metric obviously is finite at $r = 2M$; the only issue not completely obvious is whether it is invertible at $r = 2M$.
- The definition of the Christoffel symbols of the first kind is

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2} (\partial_\beta g_{\alpha\gamma} + \partial_\gamma g_{\alpha\beta} - \partial_\alpha g_{\beta\gamma})$$

Remember that the metric is a tensor and therefore transforms under coordinate change as any other tensor does.

- This is a very short exercise (it should take less than a minute to write down the answers). However, be careful before answering the question — it may be tempting to give a short but incorrect answer.