Daniel Grumiller November 2^{nd} 2020

Gravity and holography in lower dimensions I

(4.1) **OPE**
Derive the OPE

$$R\big(T(z)\phi(w,\bar{w})\big) = \frac{h}{(z-w)^2}\,\phi(w,\,\bar{w}) + \frac{1}{z-w}\,\partial_w\phi(w,\bar{w}) + \dots$$

between the holomorphic part of the stress tensor $T(z)$ and a conformal primary $\phi(w,\,\bar{w})$ of weights $(h,\bar{h})$ using the relations

$$\delta_\varepsilon\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{\mathcal{C}(w)}\mathrm{d}z\,\varepsilon(z)R\big(T(z)\phi(w,\,\bar{w})\big) + \text{barred}$$

$$\delta_\varepsilon\phi(w,\bar{w}) = h\,\partial_w\varepsilon(w)\,\phi(w,\,\bar{w}) + \varepsilon(w)\partial_w\phi(w,\,\bar{w}) + \text{barred}$$

where the ellipsis refers to the barred sector as well as to regular terms (without poles) and R denotes radial ordering as defined in the lectures.

(4.2) **Global conformal transformations**
Consider the Virasoro generators $L_n = -z^{n+1}\partial_z$ and $\bar{L}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$ for $n = \pm 1, 0$ and show that they form an $\mathrm{sl}(2,\,\mathbb{R})\oplus\mathrm{sl}(2,\,\mathbb{R})$ subalgebra of Vir$\oplus$ Vir. Take polar coordinates $z = r\,e^{i\varphi}$ with $r \in (0,\infty)$ and $\varphi \sim \varphi + 2\pi$ and express these six generators in terms of these new coordinates. Check in particular that the Hamiltonian $H = L_0 + \bar{L}_0$ generates radial dilatations and $P = i(L_0 - \bar{L}_0)$ angular rotations.

(4.3) **Commutation relation of primary fields with stress tensor**
Using the definition of the stress tensor in terms of Virasoro modes

$$T(z) = \sum_n z^{-n-2}\,L_n$$

and the definition of a (chiral) primary field of weight h in terms of modes

$$\phi(z) = \sum_n z^{-m-h}\phi_m$$

as well as the OPE displayed above verify the commutator

$$[L_n,\,\phi_m] = \big((h-1)n - m\big)\,\phi_{n+m}\,.$$

These exercises are due on November 17$^{\text{th}}$ 2020.

Hints:

- It is sufficient to derive results for one holomorphic sector. Express the quantities on the right hand side of the last equality as contour integrals and compare them with the right hand side of the penultimate equality.

- For the first part calculate all the Lie brackets and show that the structure constants are precisely the one of sl(2, $\mathbb{R}$) for each chiral sector, e.g. $[L_1, L_{-1}] = 2L_0$ and $[L_{\pm 1}, L_0] = \pm L_{\pm 1}$. For the second part just insert the coordinate trafo $z = re^{i\varphi}$ into the definitions of L_n and its barred versions.

- The inverse relation to the one given in the exercise is

$$L_n = \frac{1}{2\pi i} \oint \mathrm{d}z \, z^{n+1} T(z)$$

where the contour encircles the origin. An analogous relation holds for ϕ_m:

$$\phi_m = \frac{1}{2\pi i} \oint \mathrm{d}w \, w^{m+h-1} \phi(w)$$

However, it may be more convenient to avoid using the mode expansion for $\phi(w)$ and calculate first the commutators $[L_n, \phi(w)]$ using that a radially ordered product corresponds to a commutator

$$[L_n, \phi(w)] = \frac{1}{2\pi i} \oint_w \mathrm{d}z \, z^{n+1} \, R\big(T(z)\phi(w)\big)$$

and then inserting the OPE on the right hand side. Using Cauchy's integral theorem gives you a result in terms of ϕ and $\partial_w \phi$. You should find

$$[L_n, \phi(w)] = h(n+1) w^n \phi(w) + w^{n+1} \partial_w \phi(w) \, .$$

Finally, just insert on left and right hand sides the mode expansions for $\phi(w)$ and compare corresponding Laurent coefficients in w.