Daniel Grumiller October 28th 2020

Gravity and holography in lower dimensions I

(3.1) **Special Conformal Transformations (SCTs)**
Prove that the finite form of an inversion plus translation by some constant vector b^μ plus an inversion,

$$\frac{x'^{\,\mu}}{x'^{\,2}} = \frac{x^\mu}{x^2} - b^\mu$$

is equivalent to a finite SCT generated by the same constant vector.

$$x'^{\,\mu} = \frac{x^\mu - x^2\, b^\mu}{1 - 2b^\nu x_\nu + b^2 x^2}$$

Moreover, show that finite SCTs are compatible with the infinitesimal SCTs discussed in the lectures.

$$x'^{\,\mu} = x^\mu + 2b^\nu x_\nu\, x^\mu - x^2\, b^\mu + \mathcal{O}(b^2)$$

(3.2) **CFT 2-point correlation functions**
Prove that in CFTs in at least 3 spacetime dimensions all two-point correlators vanish unless the scaling weights of both operators are the same, $\Delta_1 = \Delta_2$.

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{d_{12}}{|x_1 - x_2|^{2\Delta_1}}\,\delta_{\Delta_1,\,\Delta_2}$$

(3.3) **Unitarity bound**
Consider a state $|\Delta\rangle$ in a CFT$_d$ that has scaling dimension Δ, $D|\Delta\rangle = -i\Delta|\Delta\rangle$ and is normalized, $\langle\Delta|\Delta\rangle = 1$. Moreover, assume that this state is annihilated by special conformal transformations, $K_\mu|\Delta\rangle = 0$, and by Lorentz-trafos, $L_{\mu\nu}|\Delta\rangle = 0$. Show that the unitarity bound

$$\eta^{\mu\nu}\,\langle\Delta|K_\mu P_\nu|\Delta\rangle \geq 0$$

implies $\Delta \geq 0$. In words, unitarity demands the scaling dimension to be non-negative.

These exercises are due on November 10th 2020.

Hints:

- This should be straightforward.
- Use the results shown in the lectures, in particular

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{d_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

  and consider additionally in the general transformation formula

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x_1}^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|_{x_2}^{\Delta_2/d} \langle \phi_1(x_1')\phi_2(x_2')\rangle$$

  the action of special conformal transformations which yield a Jacobian

$$\left|\frac{\partial x'}{\partial x}\right| = \left(1 - 2b^\nu x_\nu + b^2 x^2\right)^{-d}.$$

  A useful intermediate result that you should prove is

$$|x_1' - x_2'| = \frac{|x_1 - x_2|}{\sqrt{(1 - 2b^\nu x_{1\,\nu} + b^2 x_1^2)(1 - 2b^\nu x_{2\,\nu} + b^2 x_2^2)}}$$

  which follows from the finite SCT formula

$$x'^{\,\mu} = \frac{x^\mu - x^2 b^\mu}{1 - 2b^\nu x_\nu + b^2 x^2}\,.$$

- Since K_μ annihilates $|\Delta\rangle$ you can replace $K_\mu P_\nu|\Delta\rangle$ by the commutator $[K_\mu,\, P_\nu]|\Delta\rangle$. Then just evaluate the commutator

$$[K_\mu,\, P_\nu] = 2i\left(\eta_{\mu\nu} D - L_{\mu\nu}\right)$$

  and use the properties of the state $|\Delta\rangle$ regarding D and $L_{\mu\nu}$ mentioned in the exercise.

  Note: if you use a more standard definition of dilatations, $\hat{D} = x^\mu \partial_\mu$, then the ugly factor $-i$ disappears from the definition of the scaling weight, $\hat{D}|\Delta\rangle = \Delta|\Delta\rangle$.