Daniel Grumiller November 5^{th} 2019

Black Holes I — Exercise sheet 3

(3.1) **Lorentz tensor gymnastics**
Take the following Lorentz tensor and vector

$$T_{\mu\nu} = \begin{pmatrix} -1 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 \\ 4 & 1 & 1 & 1 \end{pmatrix} \qquad k_\mu = \begin{pmatrix} 5 \\ 0 \\ 4 \\ -3 \end{pmatrix}$$

and calculate the following quantities

(a) $T^{\mu\nu}$ and $T^\mu{}_\mu$
(b) k^μ and $k_\mu k^\mu$ (is k time-, light- or spacelike?)
(c) $T_{(\mu\nu)} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right)$ and $T_{[\mu\nu]} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right)$
(d) $T_{\mu\nu}k^\nu$ and $T_{\mu\nu}k^\mu k^\nu$
(e) $T_{\mu\nu}T^{\nu\lambda}$ [using the result for (a)]

(3.2) **Euler–Lagrange equations**
Vary the following actions and write down the Euler–Lagrange equations of motion:

(a) $S = -\int \mathrm{d}t \left[q^i \dot{p}_i + H(q^i, p_i)\right]$
(b) $S = \int \mathrm{d}t \left[k_1(q)\ddot{q} - k_2(q)\dot{q} - V(q)\right]$
(c) $S = -\frac{1}{2}\int \mathrm{d}^n x \left[(\partial_\mu\phi)(\partial_\nu\phi)\eta^{\mu\nu} - m^2\phi^2 + \lambda\phi^4\right] \qquad \mu, \nu = 0, 1, \dots, (n-1)$
(d) $S = \int \mathrm{d}t\, q$

(3.3) **Minkowski metric in rotating coordinates**
Start with the Minkowski line-element

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

and introduce "rotating coordinates"

$$t' = t \qquad x' = r\,\cos\left(\phi + \omega t\right) \qquad y' = r\,\sin\left(\phi + \omega t\right) \qquad z' = z$$

where $r = \sqrt{x^2 + y^2}$ and $\phi = \arctan\left(y/x\right)$. Find the components of the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ in these coordinates, where

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x'^\mu\,\mathrm{d}x'^\nu = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

These exercises are due on November 12^{th} 2019.

Hints:

- Remember the Einstein summation convention, i.e., to sum over contracted indices. Indices are raised with the inverse Minkowski metric $\eta^{\mu\nu} = \mathrm{diag}\,(-1,1,1,1)^{\mu\nu}$ and lowered with the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}\,(-1,1,1,1)_{\mu\nu}$. Exchanging indices $T_{\mu\nu} \to T_{\nu\mu}$ amounts to transposition.

- You may drop all boundary terms/total derivative terms and use partial integrations whenever a derivative acts on a variation (you may also keep boundary terms, and you will have made your first step towards understanding D-branes). And yes, the answer you get for the equations of motion in the case (d) is really strange...

- Remember that $\mathrm{d}x'^{\mu} = \mathrm{d}x^{\nu}\,\frac{\partial x'^{\mu}}{\partial x^{\nu}}$ and insert this into the last formula to extract $g_{\mu\nu}$. You get $g^{\mu\nu}$ e.g. from taking the matrix inverse of $g_{\mu\nu}$, but this is not the only possibility.