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Gravity and holography in lower dimensions I

(3.1) **Getting into form with Hodge gymnastics**
Assuming vanishing torsion, translate the following expressions into corresponding Riemannian expressions in terms of the metric and its associated tensors (like the Riemann tensor). In all exercises A denotes a generic p-form with $0 < p < D$, where D is the spacetime dimension, and $R^a{}_b$ is the curvature 2-form. In all five cases write down the form-degree of the starting point.

(a) $\ast\,\mathrm{d}\ast\,\mathrm{d}\ast\,\mathrm{d}A$

(b) $\ast\,\mathrm{d}\ast\,\mathrm{d}\ast\,\mathrm{d}^2\ast\,\mathrm{d}\ast\,\mathrm{d}\ast A$

(c) $R^a{}_b \wedge R^b{}_a$ (for $D = 4$; a.k.a. Chern–Pontryagin form)

(d) $\epsilon^{abcd}\, R_{ab} \wedge R_{cd}$ (for $D = 4$; a.k.a. Euler form)

(e) $\omega^a{}_b\wedge R^b{}_c\wedge R^c{}_a - \frac12\,\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_d\wedge R^d{}_a + \frac{1}{10}\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_d\wedge\omega^d{}_e\wedge\omega^e{}_a$ (for $D = 5$; a.k.a. Chern–Simons form)

(3.2) **Nieh–Yan form**
Consider a four-dimensional Cartan geometry with some torsion T^a and curvature $R^a{}_b$. Prove that the top-form

$$T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b$$

is exact.

(3.3) **Cosmological Einstein–Hilbert–Palatini action**
Vary the four-dimensional action

$$I_{\textrm{cEHP}}[e^a,\,\omega^a{}_b] = \frac{1}{16\pi G}\int \epsilon_{abcd}\left(R^{ab}\wedge e^c\wedge e^d + \frac12\,\Lambda\, e^a\wedge e^b\wedge e^c\wedge e^d\right)$$

and derive the field equations. Is the corresponding theory equivalent to Einstein gravity (with cosmological constant)? What happens when boundaries of the manifold are considered?

These exercises are due on October 23rd 2018.

Hints:

- Some expressions may be zero. Whenever this applies briefly explain why the quantity vanishes. Note that we always assume metricity. The last hint is particularly useful for the last example — try to prove whether expressions like $R^b{}_c \wedge R^c{}_a$ or $\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_d$ are symmetric, anti-symmetric or neither with respect to the free indices.

- Exact means that the form can be written as $\mathrm{d}(\dots)$. Use one of the Bianchi identities. Use the Leibnitz rule.

- Variation with respect to the tetrad is straightforward, but you still need to do some work (e.g. going into index notation) to show a possible relation to the Einstein equations. Variation with respect to the spin-connection requires you to be careful with signs and partial integrations (see the last question of the exercise), but in the end it should hopefully establish a condition that we derived already in the lectures in three spacetime dimensions. A quicker way to derive the equation of motion following from varying the spin-connection is to write $R^a{}_b = (D\omega)^a{}_b$ and use partial integration for the covariant derivative.