Daniel Grumiller October 9[th] 2018

Gravity and holography in lower dimensions I

(2.1) **Generalized Fefferman–Graham expansion** [e.g. `arXiv:1310.0819`]
Consider an asymptotically AdS metric in Gaussian normal coordinates

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \gamma_{ij}\ \mathrm{d}x^i\,\mathrm{d}x^j$$

with the following asymptotic expansion in the limit of large ρ

$$\gamma_{ij} = e^{2\rho}\gamma_{ij}^{(0)} + e^{\rho}\gamma_{ij}^{(1)} + \gamma_{ij}^{(2)} + \dots$$

where $\gamma_{ij}^{(0)} = \eta_{ij}$ is the flat Minkowski metric. Calculate the asymptotic expansion for extrinsic curvature K_{ij} and its trace $K = K_{ij}\gamma^{ij}$.

(2.2) **Mass and angular momentum of BTZ** [`hep-th/9204099`]
Take the BTZ metric ($\ell = 1$, $u = t+\varphi$ and $v = t-\varphi$ where $\varphi \sim \varphi + 2\pi$)

$$\begin{aligned}\mathrm{d}s^2_{\text{BTZ}} &= \mathrm{d}\rho^2 + 4L\ \mathrm{d}u^2 + 4\bar{L}\ \mathrm{d}v^2 - \left(e^{2\rho} + 16L\bar{L}e^{-2\rho}\right)\,\mathrm{d}u\,\mathrm{d}v\\ &= \mathrm{d}\rho^2 + \left(e^{2\rho}\,\gamma_{ij}^{(0)} + \gamma_{ij}^{(2)} + \dots\right)\,\mathrm{d}x^i\,\mathrm{d}x^j\end{aligned}$$

and calculate the holographically renormalized Brown–York stress tensor ($G_N = 1$)

$$T_{ij}^{\text{BY}} = \frac{1}{8\pi}\left(\gamma_{ij}^{(2)} - \gamma_{ij}^{(0)}\,\mathrm{Tr}\,\gamma^{(2)}\right)$$

for all values of $m = L + \bar{L}$ and $j = L - \bar{L}$. Derive expressions for the conserved mass $M = \oint \mathrm{d}\varphi\, T_{tt}^{\text{BY}}$ and angular momentum $J = \oint \mathrm{d}\varphi\, T_{t\varphi}^{\text{BY}}$. Which solution do you obtain for the special case $M = -1/8$, $J = 0$?

(2.3) **Asymptotic Killing vectors of asymptotically AdS$_3$**
Start with the Fefferman–Graham expansion

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \left(e^{2\rho/\ell}\,\eta_{ij} + \gamma_{ij}^{(2)}\right)\,\mathrm{d}x^i\,\mathrm{d}x^j$$

where η_{ij} is the 2-dimensional Minkwoski metric, e.g. in light-cone gauge. Derive all asymptotic Killing vectors for the following two sets of boundary conditions:

(a) $\gamma_{ij}^{(2)} = \mathcal{O}(1)_{ij} + \mathcal{O}(e^{-2\rho/\ell})_{ij} = \delta\gamma_{ij}^{(2)}$
(b) $\gamma_{ij}^{(2)} = 0 = \delta\gamma_{ij}^{(2)}$

Is there some set of conditions on $\gamma_{ij}^{(2)}$ where you get fewer asymptotic Killing vectors than in (b)?

These exercises are due on October 16[th] 2018.

Hints:

- Calculate first the expansion for the inverse metric

$$\gamma^{ij} = e^{-2\rho}\,\hat{\gamma}^{ij}_{(0)} + e^{-3\rho}\,\hat{\gamma}^{ij}_{(1)} + e^{-4\rho}\,\hat{\gamma}^{ij}_{(2)} + \dots$$

  and determine the coefficients $\hat{\gamma}^{ij}_{(n)}$ by requiring $\gamma^{ij}\gamma_{jk} = \delta^i_k$. It is convenient to use conventions such that all boundary indices are raised and lowered with the flat metric $\gamma^{(0)}_{ij} = \eta_{ij}$. Be careful with signs and be sure that you take into account all terms, particularly in $\hat{\gamma}^{ij}_{(2)}$. Obtaining the extrinsic curvature tensor is straightforward since we are in Gaussian normal coordinates (see your lecture notes or re-derive the result for K_{ij} in Gaussian normal coordinates). The trace K is then obtained from multiplying the expansions $K_{ij}\gamma^{ij}$ up to the required order.

- Read off the Fefferman–Graham expansion matrices $\gamma^{(0)}_{ij}$ and $\gamma^{(2)}_{ij}$. Insert them into the result for the Brown–York stress tensor and give explicitly expressions for T_{uu}, T_{vv} and T_{uv}, either in terms of (m, j) or in terms of $(L, \bar{L})$. Then calculate the conserved charges using the coordinate transformation $(u, v) \to (t, \varphi)$ and the standard tensor transformation law. To study the special case $M = -1/8$ and $J = 0$ just insert the corresponding values for $L, \bar{L}$ into the line-element. After a suitable shift $\rho \to \rho + \rho_0$ and upon expressing (u, v) in terms of (t, φ) it should be manifest which spacetime this is [it has a lot of Killing vectors]. Avoid the trap of incorrectly evaluating the trace — the trace is defined by contraction with the boundary metric $\gamma^{(0)\,ij}$.

- Part (a) is just a recap of calculations we did in Black Holes II, but since the result is important it does not hurt to recap. Part (b) is simpler, but since we did not do it yet you actually have to do the calculations. Just apply the defining property of asymptotic Killing vectors, Eq. (1) in section 2.1 of the lecture notes for these lectures, and remember that typically a subset of the asymptotic Killing vectors are actual Killing vectors or conformal Killing vectors, defined by $\mathcal{L}_\xi g_{ij} \propto g_{ij}$. [For cross-checks: your result for the number of asymptotic Killing vectors should be infinitely many for (a) and six for (b).] If you carefully did part (b) you should have some idea how to answer the last question (note that you do not have to provide a proof for your last answer, though of course you are encouraged to do so).