Daniel Grumiller October 9th 2020

Gravity and holography in lower dimensions I

(1.1) **Dimension as parameter**
Discuss the possibilities for theoretical calculations that could emerge when considering General Relativity in D spacetime dimension, keeping D arbitrary (potentially allowing for analytic continuation to non-integer D). Focus then specifically on the Schwarzschild–Tangherlini solution (where $r_h = \text{const.}$ is the horizon radius)

$$\mathrm{d}s^2 = -\left(1 - \frac{r_h^{D-3}}{r^{D-3}}\right)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{r_h^{D-3}}{r^{D-3}}} + r^2\,\mathrm{d}\Omega^2_{S^{D-2}}$$

and discuss some of its main features, in particular for $D \to \infty$, $D \to 3$ and $D \to 2$. (As usual, $\mathrm{d}\Omega^2_{S^{D-2}}$ denotes the line-element of the round $(D-2)$-sphere).

(1.2) **Isometries of metric with $(-,-,+,+)$ signature**
Derive all isometries of the metric $\mathrm{d}s^2 = -\mathrm{d}t^2 - \mathrm{d}w^2 + \mathrm{d}x^2 + \mathrm{d}y^2$ by solving the Killing equation. Is this metric maximally symmetric? Consider the subalgebra of the isometry algebra that additionally preserves the hyperboloid $-t^2 - w^2 + x^2 + y^2 = -\ell^2 = \text{const.}$ What is the Lie bracket algebra formed by the Killing vectors of this subalgebra?

(1.3) **Reminder of variational principle for Euclidean AdS$_3$**
Show that the (Euclidean) action

$$\Gamma = -\frac{1}{16\pi G}\int \mathrm{d}^3x\sqrt{g}\left(R + \frac{2}{\ell^2}\right) - \frac{1}{8\pi G}\int \mathrm{d}^2x\sqrt{\gamma}\left(\alpha K + \frac{\beta}{\ell}\right)$$

with the boundary conditions

$$\begin{aligned}
g_{rr} &= \frac{\ell^2}{r^2} + \mathcal{O}(1/r^4) &\qquad g_{rt} &= \mathcal{O}(1/r^3)\\
g_{tt} &= \frac{r^2}{\ell^2} + \mathcal{O}(1) &\qquad g_{r\varphi} &= \mathcal{O}(1/r^3)\\
g_{\varphi\varphi} &= r^2 + \mathcal{O}(1) &\qquad g_{t\varphi} &= \mathcal{O}(1) + O(1/r)
\end{aligned}$$

for the metric has a well-defined variational principle only if $2\alpha = 1 - \beta$.

These exercises are due on October 20th 2020.

Hints/comments:

- Regarding the first part: when D is arbitrarily large, $1/D$ is arbitrarily small. Consider in which circumstances it can be useful in theoretical physics to have a small parameter. Concerning the second part, check what happens with typical gradients as $D \to \infty$, derive what happens to the Killing norm as $D \to 3$ (is the limit unique?) and give some physical interpretation of the force law you find for $D \to 2$; in particular, do you have the expected Coulomb-like behavior and if so, does this correspond to a fall-off behavior or a confining behavior?

- For the first part let me remind you of the Killing equation

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu \xi^\alpha + g_{\nu\alpha}\partial_\mu \xi^\alpha = 0$$

  and that 'maximally symmetric' means there are $D(D+1)/2$ linearly independent Killing vectors, where in our case $D = 4$. For the second part you can either use the analogy to special relativity in four spacetime dimensions (only one sign changes as compared to special relatvity) and directly write down the result for the required subalgebra, or you check explicitly which Killing vectors preserve the hyperboloid and then calculate all necessary Lie brackets to determine the structure constants of the required subalgebra. Note that the hyperboloid equation defines anti-de Sitter space in three spacetime dimensions with AdS radius ℓ.

- Recall that "well-defined variational principle" is synonymous for "the first variation of the full action vanishes on-shell (including all boundary contributions) for all variations that preserve the specified boundary conditions". Solving this exercise is lengthy, but useful — not just to build your character, but to recollect some of the required tools from Black Holes II. If you get stuck you find guidance in Eqs. (1)-(14) of `1402.3687`. For comparison, here are the results for normal vector n_μ, induced volumes form $\sqrt{\gamma}$ and trace of extrinsic curvature K:

$$n_\mu = \delta^r_\mu \frac{\ell}{r} + \mathcal{O}(1/r^3)$$
$$\sqrt{\gamma} = \frac{r^2}{\ell} + \mathcal{O}(1)$$
$$K = \frac{2}{\ell} + \mathcal{O}(1/r^2)$$