Black Holes II — Exercise sheet 6

(16.1) **High-frequency limit of Regge–Wheeler equation**
The Regge–Wheeler equation can be approximated by a confluent hypergeometric equation in the high-frequency limit $\omega M \to \infty$. Solving this equation one can show that the asymptotic amplitudes appearing in the solution ingoing on the horizon $\hat{\psi}_l(r_*, \omega) \sim e^{-i\omega r_*}$ for $r_* \to -\infty$ and behaving as $\hat{\psi}_l(r_*, \omega) = A_{\text{out}}(\omega)\, e^{i\omega r_*} + A_{\text{in}}(\omega)\, e^{-i\omega r_*}$ for $r_* \to \infty$ at infinity are given by

$$A_{\text{out}} \approx \frac{\Gamma(1-4i\omega M)}{\sqrt{2\pi}(4i\omega M)^{1/2-4i\omega M}}\, e^{-4i\omega M} \qquad A_{\text{in}} \approx \frac{i\Gamma(1-4i\omega M)}{\Gamma(1/2-4i\omega M)\sqrt{4i\omega M}}\,.$$

Use Stirling's formula to show that the reflection coefficient A_R in the high-frequency limit $\omega M \to \infty$ behaves as

$$|A_R|^2 \approx e^{-8\pi\omega M}\,.$$

Interpret this result concisely.

(16.2) **Asymptotic quasi-normal modes**
Take a two-dimensional dilaton gravity action coupled non-minimally to a massless scalar field ϕ

$$S = S^{\text{2DG}} + S^{\text{mat}} \qquad S^{\text{mat}} = -\frac{1}{2}\int \mathrm{d}^2x\sqrt{-g}\, X^p\, (\nabla\phi)^2$$

and

$$S^{\text{2DG}} = \frac{1}{\kappa}\int \mathrm{d}^2x\sqrt{-g}\,\Big[XR - U(X)(\nabla X)^2 - 2V(X)\Big]$$

with constant p. It can be shown that for suitable choices of the functions U, V and for $p=1$ the model above is classically equivalent to the s-wave part of General Relativity minimally coupled to a massless free scalar field, see e.g. hep-th/0604049. Using the monodromy approach by Motl and Neitzke (gr-qc/0212096 and hep-th/0301173), the quasi-normal mode spectrum of this model was analyzed in gr-qc/0408042 in the limit of large damping. For the complex frequency ω the asymptotic relation

$$e^{\omega/T_H} = -\big(1 + 2\cos\left(\pi(1-p)\right)\big)$$

was found, where T_H is the Hawking temperature of the black hole. Asymptotic means that the imaginary part of ω/T_H is large and positive. Consider the minimally coupled case ($p=0$) and the Schwarzschild case ($p=1$) and derive formulas for the real and imaginary parts of ω/T_H. Compare the $p=1$ case with the (computer-) experimental results for the Schwarzschild black hole by Nollert (*Phys. Rev.* **D47** (1993) 5253) and Andersson (*Class. Quant. Grav.* **L10** (1993) 61) who found the asymptotic formula

$$\omega M \approx 0.0437 + \frac{i}{4}\,\big(n + \frac{1}{2}\big)$$

Does the 2D dilaton gravity formulation of the Schwarzschild black hole lead to the correct asymptotic spectrum for quasi-normal modes?

(16.3) **Superradiant scattering**
Derive Eq. (25) of the lecture notes from section 7.3.

These exercises are due on May 19$^{\text{th}}$ 2020.

Hints:

- Remember the standard definition of the reflection coefficient (or derive it by comparison with the definitions given in the lecture notes)

$$A_R = \frac{A_{\text{out}}}{A_{\text{in}}}\,.$$

For your convenience, the Stirling formula for complex z can be presented as

$$\ln\Gamma(z) = (z - \frac{1}{2})\,\ln z - z + \frac{\ln 2\pi}{2} + \mathcal{O}(1/z)$$

provided the real part of z is positive. Note that $\ln(-1 - i\varepsilon) = -i\pi + i\varepsilon + \mathcal{O}(\varepsilon^2)$ for small but positive ε.
If you are really brave/have a lot of time/have a loving relationship with confluent hypergeometric functions then try to derive the amplitude formula given in the exercise instruction.

- The longest part of this exercise is reading it. You only have to know that $T_H = 1/(8\pi M)$ for the Schwarzschild black hole. We motivated this result last semester, see exercise (9.3); we shall come back to black hole thermodynamics in the following lectures.

- The shortest part of this exercise is reading it. Yet, it is not too long. Just remember that according to the text in section 7.3 the asymptotic behavior of the infalling modes $\hat{\psi}_{lm}$ (labelled by angular quantum number l and magnetic quantum number m) is given by

$$\lim_{r_*\to-\infty}\hat{\psi}_{lm} = e^{-i(\omega - m\Omega)r_*} \qquad \lim_{r_*\to\infty}\hat{\psi}_{lm} = \left(A_R(\omega)e^{i\omega r_*} + e^{-i\omega r_*}\right)\frac{1}{A_T(\omega)}\,.$$

The complex conjugate of the above, $\hat{\psi}^*_{lm}$, is a second, linearly independent solution to the Teukolsky equation. Exploit constancy of the Wronskian, $W_{lm} = \hat{\psi}_{lm}\hat{\psi}^{*\prime}_{lm} - \hat{\psi}^*_{lm}\hat{\psi}'_{lm}$, to derive the desired result.