Daniel Grumiller May 4^{th} 2020

Black Holes II — Exercise sheet 5

(15.1) **Massless spin-2 propagator in four spacetime dimensions**
Derive the massless spin-2 propagator from massless polarization sums (possibly up to terms that vanish when transversality is imposed).

(15.2) **Energy loss through gravitational waves and binary systems**
Argue that the power radiated in gravitational waves in the weak field and small velocity approximation is given by

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} \propto \langle \dddot{q}_{ij}(t)\,\dddot{q}^{ij}(t)\rangle$$

where $q_{ij}(t) = Q_{ij}(t) - \frac{1}{3}\,\delta_{ij}\,Q^k_k(t)$ with the quadrupole moment of the source $Q^{ij}(t) = \int \mathrm{d}^3x'\,x'^i x'^j\,T^{00}(t,\,\vec{x}')$ and $\langle\rangle$ denotes time-average. Check or argue that for binary systems of two equal masses M at distance R from each other rotating in the plane $z = 0$ with angular velocity ω the quadrupole moment (possibly up to an overall factor) is given by $Q_{xx} = MR^2\cos^2(\omega t)$, $Q_{xy} = MR^2\cos(\omega t)\sin(\omega t)$, $Q_{yy} = MR^2\sin^2(\omega t)$. Combine these results to prove the gravitational wave luminosity formula

$$P \propto \Big(\frac{M}{T}\Big)^{10/3}$$

where $T \propto 1/\omega$ is the period. Finally, derive the (dimensionless) rate at which the period decreases,

$$\frac{\mathrm{d}T}{\mathrm{d}t} \propto -\Big(\frac{M}{T}\Big)^{5/3}$$

and answer what happens to the radius of a binary system, such as the Hulse–Taylor pulsar or a black hole binary, as it continues to emit gravitational radiation.

(15.3) **Gravitational collapse and gravitational wave emission**
Consider some spherically symmetric mass distribution in vacuum (without cosmological constant, i.e., asymptotically flat) of total mass M that undergoes gravitational collapse to a black hole of mass $m < M$. Assume that the distribution maintains spherical symmetry at all times. What is the maximal amount of gravitational radiation (expressed in terms of M and m) that can be observed by an observer far from the center?

These exercises are due on May 12^{th} 2020.

Hints:

- Use the +-and ×-polarization tensors $\epsilon^{+}_{\mu\nu}$ and $\epsilon^{\times}_{\mu\nu}$ for a gravitational wave travelling in z-direction with speed of light. Exploit Lorentz-invariance to argue that the sum over polarizations must be given by

  $$\sum_{I=+,\times} \epsilon^I_{\mu\nu}\epsilon^I_{\alpha\beta} = A\big(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}\big)+B\eta_{\mu\nu}\eta_{\alpha\beta}+k_\lambda\text{-terms}(\lambda=\mu,\nu,\alpha,\beta)\,.$$

  Use the explicit results for the +-and ×-polarization tensors $\epsilon^{+}_{\mu\nu}$ and $\epsilon^{\times}_{\mu\nu}$ to evaluate these sums for $\mu=\nu=\alpha=\beta=1$ and $\mu=\alpha=1$, $\nu=\beta=2$ to determine the unknown coefficients A and B (note that possible k_λ-terms cannot contribute for $\lambda=1,2$). Finally, use arguments like in the lectures to determine the pole in the propagator.

- For the first argument you may use the analogy to electromagnetism and use the corresponding formula for the power associated with quadrupole radiation or you use dimensional arguments or you do the actual calculation (which is lengthy). For the quadrupole moment of the binary system you can simply do the calculation or use again some arguments like dimensional analysis. For the luminosity formula use the results you have already and express R in terms of T and M using Kepler's third law $R^3 \propto T^2 M$. Finally, for the dimensionless rate of period decrease you can equate the rate at which the total energy of a binary system changes, $\mathrm{d}E_{\mathrm{tot}}/\,\mathrm{d}t \propto \mathrm{d}/\,\mathrm{d}t(-M^2/R)$ to (minus) the gravitational wave luminosity and exploit again Kepler's third law to eliminate R in terms of T and M, using the approximation $\mathrm{d}M/\,\mathrm{d}t \approx 0$. *Historical note:* Long before LIGO observed directly gravitational radiation from a merging black hole binary system there was an excellent hint for the existence of gravitational radiation from the Hulse–Taylor pulsar period decrease. Both these discoveries were awarded with the Nobel prize.

- Think before you calculate.