

Black Holes II — Exercise sheet 3

(13.1) Schwarzschild black holes do not bifurcate

We saw last semester that a black hole binary system can lead to a single black hole final state, after emitting an appreciable amount of gravitational radiation in the inspiralling, merger and ringdown phases. The endstate was a Kerr black hole with mass M and angular momentum J . We also saw that a Kerr black hole is unstable against superradiance, which lowers its angular momentum and mass, while maintaining (or increasing) its horizon area. The endstate of this chain of instabilities is a Schwarzschild black hole. Prove that this endstate is classically stable against bifurcation. In other words, show that a Schwarzschild black hole cannot decay into two (Schwarzschild or Kerr) black holes.¹

(13.2) Raychaudhuri equation for geodesic null congruences

Consider a congruence of null geodesics with tangent vector field k^a and deviation vector η^a (in 4 spacetime dimensions). Derive the analog of the Raychaudhuri equation for the expansion θ . You should find

$$k^a \nabla_a \theta = -\frac{1}{2} \theta^2 - \sigma_{ab} \sigma^{ab} + \omega_{ab} \omega^{ab} - R_{ab} k^a k^b$$

Explain why there is a factor $\frac{1}{2}$ instead of $\frac{1}{3}$.

(13.3) Prove Your Own Singularity Lemma

Given a geodesic null congruence with vanishing twist ($\omega_{ab} = 0$) and assuming the null energy condition (so that $R_{ab} k^a k^b \geq 0$) show the following lemma: If the expansion is negative, $\theta = \theta_0 < 0$, at some point on a geodesic in the congruence then $\theta \rightarrow -\infty$ along that geodesic within the affine distance $\lambda \leq 2/|\theta_0|$.

These exercises are due on April 28th 2020.

¹Unless, of course, something provides sufficient energy to trigger such a process. But when we use the word “decay” we usually mean “decay all by itself, without some external trigger”. Sidenote: the black holes that we observe have a surrounding accretion disk, which actually tends to bring the rotating black hole towards extremality; the processes described in this exercise work in the other direction and apply to black holes in vacuum.

Hints:

- Use Hawking’s area theorem. Use energy conservation. Perhaps have a look at exercises (10.2) and (10.3). This is a short exercise!
- Note that the orthogonality condition $k^a \eta_a = 0$ is not sufficient to determine uniquely η^a since the displacement vector may now have a component parallel to k^a . Thus, the displacement vector η^a orthogonal to the tangent vector k^a specifies only a co-dimension 2 parameter family of geodesics (in our 4 spacetime dimensions this is a 2-parameter family). Therefore, you must specify one further condition to fix the displacement vector η^a . It is convenient to introduce another null-vector l^a with the properties $l^a l_a = 0$, $l^a k_a = -1$ and $k^a \nabla_a l^b = 0$ (so if k^a is an ingoing null vector l^a is an outgoing one, with some convenient normalization). Then you may impose the condition

$$\eta^a l_a = 0$$

Instead of the “spatial metric” $h_{ab} = g_{ab} + t_a t_b$ introduce now the projector

$$P_{ab} = g_{ab} + k_a l_b + l_a k_b$$

that projects onto the required co-dimension 2 subspace of the tangent space, $P^a_b \eta^b = \eta^a$. The tensor field $B_{ab} = \nabla_b k_a$ must also be projected,

$$\hat{B}^a_b := P^a_c B^c_d P^d_b$$

After showing that $k^a \nabla_a \eta^b = \hat{B}^b_a \eta^a$ (remember that $k^a \nabla_a \eta^b = \eta^a \nabla_a k^b$) make the decomposition

$$\hat{B}^a_b = \frac{1}{2} \theta P^a_b + \sigma^a_b + \omega^a_b$$

and think about the correctness of the factor $\frac{1}{2}$. The rest is completely analog to the derivation of the Raychaudhuri equations for timelike geodesic congruences. This is a long exercise!

- This lemma is the light-like analog of the lemma we proved for timelike geodesic congruences during the lectures. You need the result for the Raychaudhuri equation for geodesic null congruences spelled out in exercise (13.2), and otherwise proceed analog to the lectures.