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Gravity and holography in lower dimensions I

(10.1) **Soft hairy boundary conditions**

Instead of Brown–Henneaux assume in one chiral sector the boundary conditions ($L_{\pm 1}$, L_0 are $\mathrm{sl}(2,\mathbb{R})$ generators with standard conventions)

$$A = b^{-1}\big(\mathrm{d}+a\big)b \qquad a = \big(\mathrm{d}t + \mathcal{J}(t,\,\varphi)\,\mathrm{d}\varphi\big)\,L_0 \qquad \delta a = \delta\mathcal{J}(t,\,\varphi)\,\mathrm{d}\varphi\,L_0$$

with the state-independent group element $b = e^{r/(2\ell)(L_1 - L_{-1})}$. (The other sector is analogous, up to sign changes, and does not need to be considered here.) Derive the canonical boundary charges and their asymptotic symmetry algebra. Using Fourier modes $J_n = \frac{k}{4\pi}\oint \mathrm{d}\varphi e^{in\varphi}\mathcal{J}$ find one generator J_m that commutes with all other J_n.

(10.2) **Stress-tensor two-point correlators on gravity side**

Switch on a chemical potential on the gravity side by assuming in the Chern–Simons formulation (in one chiral sector; the other one is analogous) $A = b^{-1}(\mathrm{d}+a)b$ and $b = e^{\rho L_0}$ with

$$a = \big(L_1 + \frac{\mathcal{L}(z,\bar z)}{k}L_{-1}\big)\,\mathrm{d}z + \big(-\mu(z,\bar z)L_1 + \dots\big)\,\mathrm{d}\bar z$$

where the ellipsis denotes terms completely fixed by the EOM. Localize the chemical potential, $\mu = \varepsilon\delta^{(2)}(z - z_2, \bar z - \bar z_2)$, expand the function $\mathcal{L} = \mathcal{L}^{(0)} + \varepsilon\mathcal{L}^{(1)} + \mathcal{O}(\varepsilon^2)$ with $\bar\partial\mathcal{L}^{(0)} = 0$ and solve the EOM linearized in ε. Verify that $\mathcal{L}^{(1)}$ yields the 2-point correlation function between two flux components of a CFT_2 stress tensor.

(10.3) **Flat space limit of three-dimensional gravity**

Start with the asymptotic symmetry algebra of AdS_3 Einstein gravity with Brown–Henneaux boundary conditions (here written as commutator algebra with L_0 shifted suitably)

$$[L_n^\pm,\,L_m^\pm] = (n-m)\,L_{n+m}^\pm + \frac{\ell}{8G}\,\big(n^3 - n\big)\,\delta_{n+m,\,0} \qquad\qquad [L_n^+,\,L_m^-] = 0$$

and make a change of basis

$$L_n := L_n^+ - L_{-n}^- \qquad\qquad M_n := \frac{1}{\ell}\,\big(L_n^+ + L_{-n}^-\big)\,.$$

The generators L_n, M_n still generate two copies of Virasoro for finite AdS radius ℓ, albeit in an unusual basis. Take the flat space limit $\ell \to \infty$, which leads to an İnönü–Wigner contraction of the algebra, and write down the commutation relations after you have taken this limit. The resulting algebra is the flat space limit of Virasoro.

These exercises are due on January 19th 2021.

Hints:

- Derive the boundary conditions preserving gauge transofrmations $\varepsilon = b^{-1}\hat{\varepsilon}b$; you should find $\hat{\varepsilon} = \eta(\varphi)L_0$. Then derive how the state-dependent function $\mathcal{J}$ transforms under such gauge transformations; you should find $\delta\mathcal{J} = \partial_\varphi\eta$. These results, together with the general results about charges and asymptotic symmetries in Chern–Simons, yield the canonical boundary charges and their asymptotic symmetry algebra. As a bonus you can prove charge conservation by showing that the EOM imply $\partial_t\mathcal{J} = 0$. For the last question just have a look at the algebra $[J_n, J_m] =?$ and check if there is any value for m for which the right hand side always vanishes.

  Note: you can compare with the results in `1611.09783`. *These boundary conditions were inspired by near horizon physics. The term 'soft hair' was coined by Hawking, Perry and Strominger in 2016, see* `1601.00921`.

- You should find that the EOM $\mathrm{d}a + a \wedge a = 0$ imply

  $$-\bar{\partial}\mathcal{L} = \frac{k}{2}\,\partial^3\mu + 2\mathcal{L}\partial\mu + \mu\partial\mathcal{L}$$

  which to linear order in ε yields

  $$\mathcal{L}^{(1)}(z_1, z_2) = -\frac{k}{2}\,\partial_{z_1}^4 G(z_1 - z_2, \bar{z}_1 - \bar{z}_2)$$

  where G is the Green function of the flat space Laplacean, $\partial\bar{\partial}G = \delta^{(2)}$, given by $G = \ln|z_1 - z_2|^2$; if you act on the first displayed equation above with ∂ you get the Laplacean on the left hand side and four derivatives of the δ-function on the right hand side. For comparison with the relevant CFT_2 2-point function

  $$\langle T(z_1)T(z_2)\rangle_{\text{CFT}_2} = \frac{c}{2(z_1 - z_2)^4}$$

  recall the Brown–Henneaux relation $c = 6k$.

  Note: This check of holography works for any n-point correlation function of the stress tensor with itself. See `1507.05620` *for a derivation.*

- Keep initially all terms containing the AdS radius ℓ when writing the algebra entirely in terms of the new generators L_n and M_n. Then take the limit $\ell \to \infty$. The L_n and M_n are respectively known as 'superrotations' and 'supertranslations' (no relation to supersymmetry).

  Note: The resulting algebra is known as (centrally extended) BMS_3, which is the asymptotic symmetry algebra of flat space Einstein gravity in 3D. See `gr-qc/0610130` *for a derivation of the symmetries and* `1208.1658` *for a holographic proposal based on them.*