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Gravity and holography in lower dimensions I

(10.1) **Log corrections to entropy from ensemble change**
Assume you have two different ensembles describing the same physical system, but with different definitions of energy, E_1 and E_2. Use the fact that the microscopic density of states transforms with a Jacobian,

$$\rho[E_2] = \frac{\partial E_1}{\partial E_2}\,\rho[E_1]$$

to derive how the entropies obtained in these two ensembles, S_1 and S_2, differ from each other. In the second part of the exercise assume that the Jacobian evaluates to some power of the entropy in the first ensemble, $\frac{\partial E_1}{\partial E_2} = a\,S_1^{-\alpha}$, with $a, \alpha > 0$. In the limit of very large entropy, $S_1 \gg 1$, how do the leading two terms look like if you express S_2 in terms of S_1?

(10.2) **Gravitons on global AdS$_3$**
Consider linearized fluctuations of the metric, $g_{\mu\nu} = g_{\mu\nu}^{\text{AdS}} + \psi_{\mu\nu}$, around global AdS,

$$\mathrm{d}s^2_{\text{AdS}} = \mathrm{d}\rho^2 - \cosh^2\rho\;\mathrm{d}t^2 + \sinh^2\rho\;\mathrm{d}\varphi^2 \qquad \varphi \sim \varphi + 2\pi\,.$$

Find all normalizable left-moving linearized fluctuations $\psi_{\mu\nu}$ that obey the SL(2)×SL(2) primary conditions $(L_1^\pm h)_{\mu\nu} = 0$ where $L_n^\pm$ are the six Killing vectors of global AdS$_3$

$$L_0^\pm = i\partial_\pm$$
$$L_{-1}^\pm = ie^{-ix^\pm}\Big(\coth(2\rho)\,\partial_\pm - \frac{1}{\sinh(2\rho)}\,\partial_\mp + \frac{i}{2}\,\partial_\rho\Big)$$
$$L_1^\pm = ie^{ix^\pm}\Big(\coth(2\rho)\,\partial_\pm - \frac{1}{\sinh(2\rho)}\,\partial_\mp - \frac{i}{2}\,\partial_\rho\Big)$$

with $x^\pm = t \pm \varphi$. By the attribute "left-moving" we mean $(L_0^-\psi)_{\mu\nu} = 0$ and $(L_0^+\psi)_{\mu\nu} = h^+\psi_{\mu\nu}$, where the weight has to be positive, $h^+ > 0$, for the mode to be called "normalizable". Work in a gauge where $\psi_{\mu-} = 0$ and exploit that ψ solving the linearized Einstein equations implies $(C_2^+ + C_2^- + 2)\psi = 0$, where $C_2^\pm = \frac{1}{2}\,(L_1^\pm L_{-1}^\pm + L_{-1}^\pm L_1^\pm) - (L_0^\pm)^2$ is the quadratic Casimir.

(10.3) **Length of geodesics on Poincaré patch AdS$_3$**
Take a constant time slice of Poincaré patch AdS$_3$ with unit AdS radius and a fixed length interval near the boundary of this slice ("near" here means that you consider not the asymptotic boundary, but rather some cut-off surface). Anchor a geodesic extending into the bulk on this fixed length interval at the cut-off surface and calculate its length as a function of the interval length and the value of the cut-off, to leading order in the cut-off.

These exercises are due on January 8th 2019.

Hints:

- Recall the microcanonical definition of entropy as the logarithm of the microscopic density, $S_{1,2} = \ln \rho[E_{1,2}]$. Actually, this is all you need to recall, the rest is straightforward.

- Applying the ancient wisdom of Fourier transforming when you do not know what else to do you can start with the separation ansatz

$$\psi_{\mu\nu}(h^+, h^-) = e^{-ih^+x^+ - ih^-x^-} \begin{pmatrix} F_{++}(\rho) & 0 & F_{+\rho}(\rho) \\ 0 & 0 & 0 \\ F_{+\rho}(\rho) & 0 & F_{\rho\rho}(\rho) \end{pmatrix}_{\mu\nu}$$

so that you work with $L_0^\pm$ eigenmodes, $L_0^\pm \psi = h^\pm \psi$, and have implemented already the required gauge conditions $\psi_{\mu-} = 0$. The left-moving condition sets one of the weights to zero, $h^- = 0$. The Einstein equations (using the quadratic Casimir) fix the (by normalizability positive!) other weight, $h^+ = 2$. The remaining steps are to solve the Killing equations corresponding to the two primary conditions, using the ansatz above. Note that some of the equations linearly combine to algebraic relations between the three functions $F_{\mu\nu}(\rho)$. One of the $++$ component equations allows to immediately determine $F_{++} \propto \tanh^2 \rho$ by simple integration. In the end this procedure yields a unique result for ψ, up to an overall factor.

- The exercise is technically simple, but you need to translate carefully all the words into formulas first. You might find a shorter way, but here I outline the way I did this. Useful coordinates for Poincaré patch AdS_3 with unit AdS radius are

$$\mathrm{d}s^2 = \frac{1}{z^2}\left(\mathrm{d}z^2 - \mathrm{d}t^2 + \mathrm{d}x^2\right).$$

The AdS_3 boundary is reached for $z \to 0$, so imposing a cut-off means instead of sending $z \to 0$ sending it to some fixed (small, but non-zero) value $z \to z_c \ll 1$. Write down the geodesic equation (either by calculating the Christoffels or by varying the geodesic action with the above metric as input; I did the latter), assume constant t and integrate the geodesic equation from some initial point $(z = z_c, t = 0, x = x_1)$ to some final point $(z = z_c, t = 0, x = x_2)$. Fix your integration constants by demanding that the length of the boundary interval, $x_1 - x_2$, takes some fixed value L. Note that you could use x as your geodesic "time", and that you have a Noether charge associated with shift symmetry $x \to x + x_0$ (that Noether charge is the Hamiltonian associated with x-translations). In fact, you can make your life a bit simpler exploiting symmetries: since we have translation invariance and the geodesic must be symmetric with respect to the midpoint of the interval, you can integrate from the initial point $(z = z_c, t = 0, x = 0)$ to the mid point $(z = z_e, t = 0, x = L/2)$. You should convince yourself that the quantity z_e has as defining property $\mathrm{d}z/\mathrm{d}\tau|_{z=z_e} = 0$, where τ is the parameter of your geodesic (for instance, $\tau = x$ or $\tau = z$). The parameter z_e is essentially the Noether charge mentioned earlier, and you should find $z_e = z\sqrt{(\mathrm{d}z/\mathrm{d}x)^2 + 1} = L/2 + \ldots$, where the ellipsis refers to terms that vanish when the cutoff is removed, $z_c \to 0$. The final task is to determine the length of the geodesic as function of L and z_c, which at this stage is straightforward. Your final result for the length should be proportional to $\ln(L/z_c)$.