Duality in 2D dilaton gravity

Based upon hep-th/0609197 with Roman Jackiw

Daniel Grumiller

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Brown University, December 2006
1. Gravity in 2D
   - Models in 2D
   - Generic dilaton gravity action

2. Vienna School Approach
   - Gravity as gauge theory
   - All classical solutions

3. Duality
   - Casimir exchange
   - Applications
Outline

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   - Applications
Riemann (Weyl+Ricci): $\frac{N^2(N^2-1)}{12}$ components in $N$ dimensions

- 4D: 20 (10 Weyl and 10 Ricci)
- 5D: 50 (35 Weyl and 15 Ricci)
- 10D: 825 (770 Weyl and 55 Ricci)
- 11D: 1210 (1144 Weyl and 66 Ricci)
- 3D: 6 (Ricci)
- 2D: 1 (Ricci scalar) $\rightarrow$ Lowest dimension with curvature
- 1D: 0

But: 2D Einstein-Hilbert: no equations of motion!

Number of graviton modes: $\frac{N(N-3)}{2}$

Stuck already in the formulation of the model? Have to go beyond Einstein-Hilbert in 2D!
Geometry in 2D
As simple as possible but not simpler...

Riemann (Weyl+Ricci): \( \frac{N^2(N^2-1)}{12} \) components in \( N \) dimensions

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Stuck already in the formulation of the model?

Have to go beyond Einstein-Hilbert in 2D!
Spherical reduction

Line element adapted to spherical symmetry:

\[ ds^2 = g^{(N)}_{\mu\nu} \, dx^\mu \, dx^\nu = g_{\alpha\beta}(x^\gamma) \, dx^\alpha \, dx^\beta - \phi^2(x^\alpha) \, d\Omega^2_{S_{N-2}}, \]

Insert into \( N \)-dimensional EH action \( I_{EH} = \kappa \int d^N x \sqrt{-g^{(N)}} R^{(N)} : \)

\[ I_{EH} = \kappa \frac{2\pi^{(N-1)/2}}{\Gamma(\frac{N-1}{2})} \int d^2 x \sqrt{-g} \phi^{N-2} \left[ R + \frac{(N - 2)(N - 3)}{\phi^2} \left( (\nabla \phi)^2 - 1 \right) \right]. \]

Cosmetic redefinition \( X \propto \left( \sqrt{\lambda} \phi \right)^{N-2} : \)

\[ I_{EH} \propto \int d^2 x \sqrt{-g} \left[ XR + \frac{N - 3}{(N - 2)X} \left( \nabla X \right)^2 - \lambda X^{(N-4)/(N-2)} \right]. \]

Scalar–tensor theory a.k.a. dilaton gravity

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Second order formulation

Similar action arises from string theory, from other kinds of dimensional reduction, from intrinsically 2D considerations, ...

Generic action:

\[ l_{2DG} = \kappa \int d^2 x \sqrt{-g} \left[ X R + U(X)(\nabla X)^2 - \lambda V(X) \right] \]  \hspace{1cm} (1)

Special case \( U = 0, V = X^2 \): EOM \( R = 2\lambda X \)

\[ l \propto \int d^2 x \sqrt{-g} R^2 \]

Similarly \( f(R) \) Lagrangians related to (1) with \( U = 0 \)

String context: \( X = e^{-2\phi} \), with \( \phi \) as string dilaton

Conformal trafo to different model with \( \tilde{U}(X) = 0 \):

\[ \tilde{V}(X) = \frac{d}{dX} w(X) := V(X)e^{Q(X)}, \text{ with } Q(X) := \int^X dy U(y) \]

\[ \underbrace{\text{conformally invariant}} \]
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conformally invariant
## Selected list of models

<table>
<thead>
<tr>
<th>Model</th>
<th>$U(X)$</th>
<th>$\lambda V(X)$</th>
<th>$\lambda w(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Schwarzschild (1916)</td>
<td>$-\frac{1}{2X}$</td>
<td>$-\lambda$</td>
<td>$-2\lambda \sqrt{X}$</td>
</tr>
<tr>
<td>2. Jackiw-Teitelboim (1984)</td>
<td>0</td>
<td>$\Lambda X^2$</td>
<td>$\frac{1}{2} \Lambda X^2$</td>
</tr>
<tr>
<td>3. Witten BH (1991)</td>
<td>$-\frac{1}{X}$</td>
<td>$-2b^2 X$</td>
<td>$-2b^2 X$</td>
</tr>
<tr>
<td>4. CGHS (1992)</td>
<td>0</td>
<td>$-2b^2$</td>
<td></td>
</tr>
<tr>
<td>5. (A)dS$_2$ ground state (1994)</td>
<td>$-\frac{a}{X}$</td>
<td>$BX$</td>
<td></td>
</tr>
<tr>
<td>6. Rindler ground state (1996)</td>
<td>$-\frac{a}{X}$</td>
<td>$BX^a$</td>
<td></td>
</tr>
<tr>
<td>7. BH attractor (2003)</td>
<td>0</td>
<td>$BX^{-1}$</td>
<td></td>
</tr>
<tr>
<td>8. SRG ($N &gt; 3$)</td>
<td>$-\frac{N-3}{(N-2)X}$</td>
<td>$-\lambda^2 X^{(N-4)/(N-2)}$</td>
<td>$-\lambda^2 \frac{N-2}{N-3}X^{(N-3)/(N-2)}$</td>
</tr>
<tr>
<td>9. All above: $ab$-family (1997)</td>
<td>$-\frac{a}{X}$</td>
<td>$BX^{a+b}$</td>
<td>$b \neq -1 : \frac{B}{b+1}X^{b+1}$</td>
</tr>
<tr>
<td>10. Liouville gravity</td>
<td>$a$</td>
<td>$be^{\alpha X}$</td>
<td></td>
</tr>
<tr>
<td>11. Reissner-Nordström (1916)</td>
<td>$-\frac{1}{2X}$</td>
<td>$-\lambda^2 + \frac{Q^2}{X}$</td>
<td>$-2\lambda^2 \sqrt{X} - \frac{2Q^2}{\sqrt{X}}$</td>
</tr>
<tr>
<td>12. Schwarzschild-(A)dS</td>
<td>$-\frac{1}{2X}$</td>
<td>$-\lambda^2 - \ell X$</td>
<td>$-2\lambda^2 \sqrt{X} - \frac{2}{3} \ell X^{3/2}$</td>
</tr>
<tr>
<td>13. Katanaev-Volovich (1986)</td>
<td>$\alpha$</td>
<td>$\beta X^2 - \Lambda$</td>
<td>$\int X e^{\alpha y} (\beta y^2 - \Lambda) dy$</td>
</tr>
<tr>
<td>14. Achucarro-Ortiz (1993)</td>
<td>0</td>
<td>$\frac{Q^2}{X} - \frac{J}{4X^3} - \Lambda X$</td>
<td>$Q^2 \ln X + \frac{J}{8X^2} - \frac{1}{2} \Lambda X^2$</td>
</tr>
<tr>
<td>15. Scattering trivial (2001)</td>
<td>generic</td>
<td>$\frac{1}{2} X(c - X^2)$</td>
<td>const. $-\frac{1}{8} (c - X^2)^2$</td>
</tr>
<tr>
<td>16. KK reduced CS (2003)</td>
<td>0</td>
<td>$-\gamma$</td>
<td>$-(1 + \sqrt{1 + \gamma^2})$</td>
</tr>
<tr>
<td>17. exact string BH (2005)</td>
<td>lengthy</td>
<td>$-\chi \prod_{i=1}^{n} (X^2 - X_i^2)$</td>
<td>lengthy $4A \cosh (X/2)$</td>
</tr>
<tr>
<td>19. KK red. conf. flat (2006)</td>
<td>$-\frac{1}{2} \tanh (X/2)$</td>
<td></td>
<td></td>
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<tr>
<td>20. 2D type 0A</td>
<td>$-\frac{1}{X}$</td>
<td>$-2b^2 X + \frac{b^2 q^2}{8\pi}$</td>
<td>$-2b^2 X + \frac{b^2 q^2}{8\pi} \ln X$</td>
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**Red:** relevant for strings

**Blue:** pioneer models

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First order formulation

Example: Jackiw-Teitelboim model \((U = 0, \lambda V = \Lambda X)\)

\[
[P_a, P_b] = \Lambda \varepsilon_{ab} J, \quad [P_a, J] = \varepsilon^b a P_b,
\]

Non-abelian BF theory:

\[
I_{BF} = \int X_A F^A = \int \left[ X_a d e^a + X_a \varepsilon^{a b} \omega \wedge e^b + X d \omega + \varepsilon_{a b} e^a \wedge e^b \Lambda X \right]
\]

field strength \(F = dA + [A, A]/2\) contains \(SO(1, 2)\) connection

\(A = e^a P_a + \omega J\), coadjoint Lagrange multipliers \(X_A\)

Generic first order action:

\[
l_{2DG} \propto \int \left[ X_a T^a + X R + \varepsilon (X^a X_a U(X) + \lambda V(X)) \right]
\]

(2)

\(T^a = d e^a + \varepsilon^{a b} \omega \wedge e^b\), \(R = d \omega\), \(\varepsilon = \varepsilon_{a b} e^a \wedge e^b\)
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Example: Jackiw-Teitelboim model ($U = 0, \lambda V = \Lambda X$)

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Field strength $F = dA + [A, A]/2$ contains $SO(1, 2)$ connection $A = e^a P_a + \omega J$, coadjoint Lagrange multipliers $X_A$

Generic first order action:

$$I_{2DG} \propto \int \left[ X_a \underbrace{T^a}_{\text{torsion}} + X \underbrace{R}_{\text{curvature}} + \varepsilon \underbrace{(X^a X_a U(X) + \lambda V(X))}_{\text{volume}} \right]$$  \hspace{1cm} (2)

$$T^a = d e^a + \varepsilon^a_{\ b} \omega \wedge e^b, \quad R = d \omega, \quad \varepsilon = \varepsilon_{ab} e^a \wedge e^b$$
Symmetries and equations of motion
Reinterpretation as Poisson-\(\sigma\) model 94: Schaller, Strobl

\[
I_{PSM} = \int \left[ dX^I \wedge A_I + \frac{1}{2} P^{IJ} A_J \wedge A_I \right]
\]

- gauge field 1-forms: \(A_I = (\omega, e_a)\), connection, Zweibeine
- target space coordinates: \(X^I = (X, X^a)\), dilaton, aux. fields
- \(P^{IJ} = -P^{JI}\), \(P^{Xa} = \varepsilon^{ab} X^b\), \(P^{ab} = \varepsilon^{ab} (\lambda V(X) + X^a X_a U(X))\)
- Jacobi: \(P^{IL} \partial_L P^{JK} + \text{perm}(IJK) = 0\)

Equations of motion (first order):
\[
dX^I + P^{IJ} A_J = 0
\]
\[
dA_I + \frac{1}{2} (\partial_I P^{JK}) A_K \wedge A_J = 0
\]

Gauge symmetries (local Lorentz and diffeos):
\[
\delta X^I = P^{IJ} \varepsilon_J
\]
\[
\delta A_I = -d\varepsilon_I - \left( \partial_I P^{JK} \right) \varepsilon_K A_J
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Duality in 2D dilaton gravity
Constant dilaton vacua:

\[ X = \text{const.}, \quad V(X) = 0, \quad R = \lambda V'(X) \]

- Minkowski, Rindler or (A)dS only
- isolated solutions (no constant of motion)

Generic solutions in EF gauge \( \omega_0 = e_0^+ = 0, \ e_0^- = 1 \):

\[
\text{d} s^2 = 2 e^{Q(X)} \text{d} u \text{d} X + e^{Q(X)} (\lambda w(X) + M) \text{d} u^2
\]

- Birkhoff theorem: at least one Killing vector \( \partial_u \)
- one constant of motion: mass \( M \)
- one parameter in action: \( \lambda \)
- dilaton is coordinate \( x^0 \) (residual gauge trasfos!)
Constant dilaton vacua and generic solutions

Light-cone components and Eddington-Finkelstein gauge (87: Polyakov, 92: Kummer, Schwarz, 96: Klösch, Strobl)

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Killing norm

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Exchange spacetime mass with reference mass

Recall

\[ ds^2 = e^{Q(X)} \left[ 2 \, du \, dX + (\lambda \, w(X) + M) \, du^2 \right] \]

Reformulate as

\[ ds^2 = e^{Q(X)} \, w(X) \left[ 2 \, du \, \frac{dX}{w(X)} + \left( M \, \frac{1}{w(X)} + \lambda \right) \, du^2 \right] \]

Leads to dual potentials

\[ \tilde{U}(\tilde{X}) = w(X) \, U(X) - e^{Q(X)} \, V(X) \]

\[ \tilde{V}(\tilde{X}) = -\frac{V(X)}{w^2(X)} \]

and to dual action

\[ \tilde{I} = \kappa \int d^2 x \sqrt{-g} \left[ \tilde{X} R + \tilde{U}(\nabla \tilde{X})^2 - M \tilde{V} \right] \]
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and to dual action

\[ \tilde{l} = \kappa \int d^2 x \sqrt{-\tilde{g}} \left[ \tilde{X} \tilde{R} + \tilde{U}(\nabla \tilde{X})^2 - M\tilde{V} \right] \]
Trick: convert parameter in action to constant of motion
Example (conformally transformed Witten BH, 92: Cangemi, Jackiw):

$$\int d^2 x \sqrt{-g}[XR - \lambda]$$

integrate in abelian gauge field ($F = * dA$)

$$\int d^2 x \sqrt{-g}[XR + YF - Y]$$

on-shell: $dY = 0$, so $Y = \lambda$
apply trick to PSM: 4D Poisson manifold, 2 Casimirs (mass, charge)
duality: exchanges mass with charge
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The *ab*-family
Schwarzschild, Jackiw-Teitelboim, ...

Useful 2-parameter family of models:

\[ U = -\frac{a}{X}, \quad V = X^{a+b} \]

Duality: \( \tilde{a} = 1 - (a - 1)/b \) and \( \tilde{b} = 1/b \)

Global structure:

\( b > 0 \): reflection at origin
here \( \xi = \ln|b| \) and \( \rho = (a - 1)/\sqrt{|b|} \)

\( b < 0 \): reflection at \( \rho \)-axis
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  - Here \( \xi = \ln |b| \) and \( \rho = (a - 1)/\sqrt{|b|} \)
- For \( b < 0 \): reflection at \( \rho \)-axis

Duality in 2D dilaton gravity
Specific case ("almost Weyl invariant"): 

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Dual model:

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(conformally transformed) Witten BH!
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Spherical reduction from $2 + \varepsilon$ to 2 dimensions:

$$U(X) = \frac{1 - \varepsilon}{\varepsilon X}, \quad V(X) = -\varepsilon(1 - \varepsilon)X^{1 - 2/\varepsilon}$$

Limit $\varepsilon \to 0$ not well-defined! No suitable rescaling of fields and coupling constants possible! [recall: in action $XR + U(X)(\nabla X)^2$]

Solution: dualize, take limit in dual formulation, dualize back

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