Physical Solutions of The Symmetric Teleparallel Gravity in Two-Dimensions

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Abstract

A 2D symmetric teleparallel gravity model is given by a generic 4-parameter action that is quadratic in the non-metricity tensor. Variational field equations are derived. For a particular choice of the coupling parameters and in a natural gauge, we give a Schwarzschild type solution and an inflationary cosmological solution.
1 Introduction

In order to investigate the physical implications of General Relativity (GR) some simple limiting cases are often sought. The low energy, static limit is attained by assuming the existence of a time-like Killing vector. However, such configurations have no dynamics. Symmetric limits at arbitrary energy scales on the other hand are obtained by assuming one or more space-like Killing vectors. For example, the assumption of spherical symmetry reduces the gravitational action upon integrating over the angular variables to an effective 2D gravity model in \((t, r)\) coordinates. Such 2D gravity models have recently been used much to discuss black hole dynamics, quantized gravity or numerical relativity [1]. It is well known that Einsteinean gravity in 2D has no dynamical degrees of freedom. One way of making this theory dynamical is to couple a dilaton scalar. It is also possible to introduce further dynamical degrees of freedom by going over to a non-Riemannian space-time by the inclusion of torsion and non-metricity. Generalized 2D gravity models with curvature and torsion and teleparallel theories with only torsion have been studied a lot [2, 3]. On the other hand 2D models including non-metricity received less attention [4, 5]. The symmetric teleparallel gravity (STPG) with zero-curvature and zero-torsion has been introduced relatively recently [6, 7, 8]. Here we consider 2D symmetric teleparallel gravity models with the most general 4-parameter action that is quadratic in the non-metricity tensor [9]. For a particular choice of parameters we have shown that the corresponding set of field equations admit both the Schwarzschild solution and an inflationary cosmological solution. It is remarkable that the STPG equivalent of Einstein gravity in 2D does not admit these solutions.

2 Mathematical Preliminaries

The triple \(\{M, g, \nabla\}\) denotes the space-time where \(M\) is a 2-dimensional differentiable manifold, \(g\) is a non-degenerate Lorentzian metric and \(\nabla\) is a linear connection. \(g\) can be written in terms of the co-frame 1-forms

\[ g = g_{\alpha\beta}dx^\alpha \otimes dx^\beta = \eta_{ab}e^a \otimes e^b \quad (1) \]

where \(\{e^a\}\) is an orthonormal co-frame and \(\{dx^a\}\) are the co-ordinate co-frame 1-forms. We take \(\eta_{ab} = \text{Diag}(-, +)\) and \(g_{\alpha\beta}\) are the co-ordinate components of the metric. The dual orthonormal frame is determined from the relations \(e^b(X_a) = \iota_a e^b = \delta^b_a\). Similarly, \(dx^\beta(\partial_a) = \iota_a dx^\beta = \delta^\beta_a\). Here \(\iota\) denotes the interior product operator that maps any \(p\)-form to a \((p - 1)\)-form. We set space-time orientation by the choice \(\epsilon_{01} = +1\) or \(*1 = e^0 \wedge e^1\).
where $*$ is the Hodge star. Finally, the connection is specified by a set of connection 1-forms \{\Lambda^a_b\}. The non-metricity 1-forms, torsion 2-forms and curvature 2-forms are defined through the Cartan structure equations:

\[
Q_{ab} = -\frac{1}{2} D\eta_{ab} = \frac{1}{2}(\Lambda_{ab} + \Lambda_{ba}) ,
\]

(2)

\[
T^a = De^a = de^a + \Lambda^e_b \wedge e^b ,
\]

(3)

\[
R^a_b = D\Lambda^a_b = d\Lambda^a_b + \Lambda^a_c \wedge \Lambda^c_b
\]

(4)

Bianchi identities follow as their integrability conditions:

\[
DQ_{ab} = \frac{1}{2}(R_{ab} + R_{ba}) ,
\]

(5)

\[
DT^a = R^a_b \wedge e^b ,
\]

(6)

\[
DR^a_b = 0
\]

(7)

If the non-metricity is non-vanishing, special attention is due in lowering and raising indices under covariant exterior derivative. We make use of the identities

\[
D * e_a = -Q \wedge * e_a + \epsilon_{ab} T^b
\]

(8)

\[
D * e_{ab} = -\epsilon_{ab} Q
\]

(9)

where $Q = \Lambda^a_a = Q^a_a$ is called the Weyl 1-form. The full connection 1-forms can be decomposed uniquely as follows [4, 5]:

\[
\Lambda^a_b = \omega^a_b + K^a_b + q^a_b + Q^a_b
\]

(10)

where $\omega^a_b$ are the Levi-Civita connection 1-forms satisfying

\[
\omega^a_b \wedge e^b = -de^a ,
\]

(11)

$K^a_b$ are the contortion 1-forms satisfying

\[
K^a_b \wedge e^b = T^a
\]

(12)

and $q^a_b$ are the anti-symmetric tensor 1-forms defined by

\[
q_{ab} = -(t_a Q_{bc}) \wedge e^c + (t_b Q_{ac}) \wedge e^c .
\]

(13)

In this decomposition the symmetric part

\[
\Lambda_{(ab)} = Q_{ab}
\]

(14)

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while the anti-symmetric part
\[ \Lambda_{[ab]} = \omega_{ab} + K_{ab} + q_{ab} . \]  

(15)

In writing down gravity models it easily becomes complicated to keep all the components of \( Q^a_b \). Therefore, people sometimes deal only with certain irreducible parts of that. To obtain the irreducible decomposition of non-metricity tensor under the 2D Lorentz group, firstly we write
\[
Q_{ab} = \overline{Q}_{ab} + \frac{1}{2} \eta_{ab} Q .
\]  

(16)

Then
\[
Q_{ab} = (1) Q_{ab} + (2) Q_{ab} + (3) Q_{ab} \]  

(17)
in terms of
\[
(2) Q_{ab} = \frac{1}{2} \left[ (\epsilon_a \Lambda) e_b + (\epsilon_b \Lambda) e_a - \eta_{ab} \Lambda \right],
\]  

(18)

\[
(3) Q_{ab} = \frac{1}{2} \eta_{ab} Q,
\]  

(19)

\[
(1) Q_{ab} = Q_{ab} - (2) Q_{ab} - (3) Q_{ab}.
\]  

(20)

where \( \Lambda := (\epsilon_b \overline{Q}_a) e^a \). These irreducible components satisfy
\[
\eta_{ab} (1) Q^a_b = \eta_{ab} (2) Q^a_b = 0 , \quad (\epsilon_a (1) Q^a_b = 0 , \quad e_a \wedge (1) Q^a_b = 0 .
\]  

(21)

Thus they are orthogonal in the following sense:
\[
(1) Q^a_b \wedge \ast (j) Q_{ab} = \delta^{ij} N_{ij} \quad \text{(no summation over } ij) \]  

(22)

where \( \delta^{ij} \) is the Kronecker symbol and \( N_{ij} \) are certain 2-forms. We calculate
\[
(1) Q^a_b \wedge \ast (1) Q_{ab} = Q^a_b \wedge \ast Q_{ab} - (2) Q^a_b \wedge \ast (2) Q_{ab} - (3) Q^a_b \wedge \ast (3) Q_{ab} ,
\]  

(23)

\[
(2) Q^a_b \wedge \ast (2) Q_{ab} = (\epsilon^a Q_{ac} (\epsilon_b Q^{hc}) \ast 1 + \frac{1}{4} Q \wedge \ast Q - (\epsilon_a Q) (\epsilon_b Q_{ab}) \ast 1 ,
\]  

(23)

\[
(3) Q^a_b \wedge \ast (3) Q_{ab} = \frac{1}{2} Q \wedge \ast Q .
\]  

(23)
3 Symmetric Teleparallel Gravity

We formulate STPG by a variational principle from an action

$$I = \int_M \left( L + R^a_b \rho^b_a + T^a \lambda_a \right)$$

(24)

where $L$ is a Lagrangian density 2-form quadratic in the non-metricity tensor, $\rho^b_a$ and $\lambda_a$ are the Lagrange multiplier 0-forms imposing the constraints

$$R^a_b = 0 \quad , \quad T^a = 0.$$  

(25)

The gravitational field equations are derived from (24) by independent variations with respect to the connection $\{\Lambda^a_b\}$ and the orthonormal co-frame $\{e^a\}$ (and the Lagrange multipliers):

$$\lambda_a \wedge e^b + D\rho^b_a = -\Sigma^b_a,$$  

(26)

$$D\lambda_a = -\tau_a.$$  

(27)

where $\Sigma^b_a = \frac{\partial L}{\partial \Lambda^a_b}$ and $\tau_a = \frac{\partial L}{\partial e^a}$. In principle (26) is solved for the Lagrange multipliers $\lambda_a$ and $\rho^b_a$ and substituted in (27) that governs the dynamics of the gravitational fields. It is important to notice that $D\lambda_a$ rather than the Lagrange multipliers themselves appear in (27). Thus we must be calculating $D\lambda_a$ directly and that we can do by taking the covariant exterior derivative of (26):

$$D\lambda_a \wedge e^b = -D\Sigma^b_a.$$  

(28)

We used above the constraints

$$T^a = 0 \quad , \quad D^2 \rho^b_a = R^b_c \wedge \rho^c_a - R^c_a \wedge \rho^b_c = 0$$

(29)

where the covariant exterior derivative of a (1,1)-type tensor is given by

$$D\rho^b_a = d\rho^b_a + \Lambda^b_c \wedge \rho^c_a - \Lambda^c_a \wedge \rho^b_c.$$  

(30)

Finally we arrive at the gravitational field equations

$$D\Sigma^b_a - \tau_a \wedge e^b = 0.$$  

(31)

We now write down the most general Lagrangian density 2-form which is quadratic in the non-metricity tensor [9]:

$$L = \frac{1}{2\kappa} \left[ \sum_{I=1}^3 k_I (1) Q_{ab} \wedge * (1) Q^{ab} + k_4 \left( (2) Q_{ab} \wedge e^b \right) \wedge * \left( (3) Q^{bc} \wedge e^c \right) \right].$$  

(32)
where \( k_1, k_2, k_3, k_4 \) are dimensionless coupling constants and we also introduced \( \kappa = \frac{8\pi G}{c^3} \) with \( G \) being Newton’s gravitational constant. Inserting (23) into (32) we find
\[
L = \frac{1}{2\kappa} \left[ c_1 Q_{ab} \wedge *Q^{ab} + c_2 (i_a Q^{ac})(i^b Q_{bc}) * 1 + c_3 Q \wedge *Q + c_4 (i_a Q)(i_b Q^{ab}) * 1 \right] \tag{33}
\]
where we defined new coupling constants:
\[
c_1 = k_1, \quad c_2 = -k_1 + k_2, \quad c_3 = -\frac{3}{4} k_1 + \frac{1}{4} k_2 + \frac{1}{2} k_3 + \frac{1}{4} k_4, \quad c_4 = k_1 - k_2 - \frac{1}{2} k_4. \tag{34}
\]

We remark at this point that STP equivalent of the Einstein-Hilbert Lagrangian density will be obtained for the choice \( c_1 = -1, \ c_2 = 0, \ c_3 = -1, \ c_4 = 2 \). We wish to show this briefly. Firstly we use the decomposition of the full connection (10), with \( K_{ab} = 0, \ \Lambda_{ab} = \omega_{ab} + \Omega_{ab} \)
\[
R_{ab}(\Lambda) = d\Lambda_{ab} + \Lambda_{ac} \wedge \Lambda_{cb} \tag{35}
\]
By substituting that into \( R^a_b(\Lambda) \) we decompose the non-Riemannian curvature as the follows:
\[
R^a_b(\Lambda) = d\Lambda^a_b + \Lambda^a_c \wedge \Lambda^c_b = R^a_b(\omega) + D(\omega)\Omega^a_b + \Omega^a_c \wedge \Omega^c_b. \tag{36}
\]
Here \( R^a_b(\omega) \) is the Riemannian curvature 2-form and \( D(\omega) \) is the covariant exterior derivative with respect to the Levi-Civita connection. To set \( R^a_b(\Lambda) = 0 \) for STP space-time yields the Einstein-Hilbert Lagrangian 2-form
\[
L_{EH} = R^a_b(\omega) \wedge *e^b_a = \left[ D(\omega)\Omega^a_b \right] \wedge *e^b_a - \Omega^a_c \wedge \Omega^c_b \wedge *e^b_a \tag{37}
\]
Here after using the equality
\[
d(\Omega^a_b \wedge *e^b_a) = [D(\omega)\Omega^a_b] \wedge *e^b_a - \Omega^a_b \wedge [D(\omega) * e^b_a] \tag{38}
\]
we discard the exact form and notice $D(\omega) * e_a^b = 0$ because $T^a(\omega) = 0$ and $Q^a_b(\omega) = 0$ (see eq. (9)). Thus, up to a closed form

$$L_{EH} = \frac{1}{2\kappa} \Omega^a_c \wedge \Omega^c_b \wedge *e_a^b$$

$$= \frac{1}{2\kappa} (Q^{ac} + q^{ac}) \wedge (Q_{cb} + q_{cb}) \wedge *e_a^b$$

$$= \frac{1}{2\kappa} (Q^{ac} \wedge Q_{cb} + q^{ac} \wedge q_{cb}) \wedge *e_a^b$$

$$= \frac{1}{2\kappa} [-Q_{ab} \wedge *Q^{ab} - Q \wedge *Q + 2(t_b Q)(t_a Q^{ab}) * 1] \quad (39)$$

where $\kappa$ is gravitational coupling constant.

### 4 Solutions

We obtain the following contributions to the variational field equations (31) coming from (33):

$$\Sigma^b_a = \sum_{i=0}^4 c_i \Sigma^b_a \quad , \quad \tau_a = \sum_{i=0}^4 c_i \tau_a \quad (40)$$

where

$$\begin{align*}
1\Sigma_{ab} &= 2 * Q_{ab} , \quad (41) \\
2\Sigma_{ab} &= \ell^e Q_{ac} * e_b + \ell^e Q_{bc} * e_a , \\
3\Sigma_{ab} &= 2\eta_{ab} * Q , \quad (43) \\
4\Sigma_{ab} &= \frac{1}{2} (t_a Q) * e_b + \frac{1}{2} (t_b Q) * e_a + \eta_{ab}(t_c Q^{cd}) * e_d , \quad (44) \\
1\tau_a &= -(t_a Q^{bc}) \wedge *Q_{bc} - Q^{bc} \wedge (t_a * Q_{bc}) , \quad (45) \\
2\tau_a &= (t_b Q^{bc})(t^d Q_{cd}) * e_a - 2(t_a Q^{bd})(t^e Q_{cd}) * e_b , \quad (46) \\
3\tau_a &= -(t_a Q) \wedge *Q - Q \wedge (t_a * Q) , \quad (47) \\
4\tau_a &= (t_b Q)(t_c Q^{bc}) * e_a - (t_a Q)(t_c Q^{bc}) * e_b - (t_b Q)(t_a Q^{bc}) * e_c . \quad (48)
\end{align*}$$

One can consult Ref.[8] for the details of variations.

A generic class of solutions will be obtained in the coordinate frame $e^a = dx^a$ in which $\Lambda^a_{\beta} = 0$. We call this the natural or inertial gauge choice:

$$R^a_{\beta} = d\Lambda^a_{\beta} + \Lambda^a_{\gamma} \wedge \Lambda^\gamma_{\beta} = 0 , \quad (49)$$

$$T^a = d(dx^a) + \Lambda^a_{\beta} \wedge dx^\beta = 0 , \quad (50)$$

$$Q_{\alpha\beta} = -\frac{1}{2} D g_{\alpha\beta} = -\frac{1}{2} d g_{\alpha\beta} \neq 0 \quad . \quad (51)$$
After a frame transformation via the zweibein \( e^\alpha = h^a_\alpha dx^\alpha \) and setting \( \Lambda^a_b = h^a_\alpha \Lambda^\alpha_\beta h^\beta_b + h^a_\alpha dh^\alpha_b \) we obtain the orthonormal components

\[
R^a_b = h^a_\alpha R^\alpha_\beta h^\beta_b = 0, \quad T^a = h^a_\alpha T^\alpha = 0, \quad Q_{ab} = Q_{a\beta} h^\alpha_a h^\beta_b \neq 0.
\]  

This shows that in the natural gauge, the field equations may be solved just by a metric ansatz.

4.1 Static solutions

First we consider

\[
g = -f^2(r)dt^2 + g^2(r)dr^2.
\]  

The co-ordinate components of the metric and the zweibein reads, respectively, as follows

\[
g_{\alpha\beta} = \begin{pmatrix} -f^2 & 0 \\ 0 & g^2 \end{pmatrix}, \quad h^a_\alpha = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}.
\]  

The coordinate components of non-metricity is found to be

\[
Q_{\alpha\beta} = -\frac{1}{2}dg_{\alpha\beta} = \begin{pmatrix} f' e^1 & 0 \\ 0 & -g' e^1 \end{pmatrix}
\]  

where prime denotes the derivative with respect to \( r \). The orthonormal components are obtained through a frame transformation

\[
Q_{a\beta} = Q_{a\beta} h^\alpha_a h^\beta_b = \begin{pmatrix} f' e^1 & 0 \\ 0 & -g' e^1 \end{pmatrix}.
\]  

In the above configuration the only non-trivial field equation comes from the trace of (31):

\[
d\Sigma^{a}_a = 0
\]  

that reads explicitly

\[
\alpha \left[ \left( \frac{f'}{f} \right)^2 + \left( \frac{f'}{f} \right)^2 \right] + \beta \left[ \left( \frac{g'}{g} \right)^2 - \left( \frac{g'}{g} \right)^2 \right] + (\beta - \alpha) \frac{f'}{f} \frac{g'}{g} = 0
\]  

where \( \alpha = 2c_1 + 4c_3 + c_4 \), \( \beta = 2c_1 + 2c_2 + 4c_3 + 3c_4 \). Mathematically, this equation has infinitely many solutions because there are two functions and
only one equation. That is, given \( g(r) \) we determine the corresponding \( f(r) \). Physically, however, due to the observational success of the Schwarzschild solution of GR, we seek solutions of the form

\[
f(r) = \left( a + \frac{b}{r} \right)^p, \quad g(r) = \left( a + \frac{b}{r} \right)^q.
\] (59)

Then (58) becomes

\[
(p - q + 1 + 2ar) (p\alpha + q\beta) = 0
\] (60)

and therefore we let \( \beta = -\frac{p}{q} \alpha \). We note here that STP Einstein-Hilbert action corresponds to the choice \( p = 0 \) with arbitrary \( q \). Finally with \( p = 1/2 \), \( \beta = \alpha \) and a suitable choice of \( a \) and \( b \), the Schwarzschild metric is obtained:

\[
g = -(1 - \frac{2m}{r}) dt^2 + \frac{dr^2}{(1 - \frac{2m}{r})}.
\] (61)

### 4.2 Cosmological solutions

For cosmological solutions we start with the metric

\[
g = -dt^2 + R^2(t) dr^2
\] (62)

where \( R(t) \) is the expansion function. In the natural gauge, the non-metricity has just one non-zero component

\[
Q_{11} = -\frac{\dot{R}}{R} e^0, \quad \text{others} = 0
\] (63)

where dot denotes the derivative with respect to \( t \). For this configuration, the only non-trivial part of (31) is, again, its trace; \( d\Sigma_a ^a = 0 \) which reads

\[
\alpha \left[ \left( \frac{\dot{R}}{R} \right)^2 + \left( \frac{\dot{R}}{R} \right)^2 \right] = 0.
\] (64)

This equation accepts two classes of solution, as well.

1. For \( \alpha \neq 0 \), we obtain

\[
R(t) = a + bt
\] (65)

where \( a \) and \( b \) integration constants. The case of STP Einstein-Hilbert action belongs here.
2. For \( \alpha = 2c_1 + 4c_3 + c_4 = 2k_3 + \frac{1}{2}k_4 = 0 \), we set

\[
\left( \frac{\dot{R}}{R} \right)^2 + \left( \frac{\dot{R}}{R} \right)^2 = S(t)
\] (66)

where \( S(t) \) is any \( t \) dependent function. This result contains many familiar solutions. For example;

- For \( S = H^2 \): Hubble constant, we obtain the Robertson-Walker metric satisfying the perfect cosmological principle. A coordinate transformation \( \tan \frac{\sqrt{\kappa} r}{\sqrt{\kappa} \rho} = \frac{\sqrt{\kappa}}{\sqrt{\kappa} \rho} \), takes the metric to

\[
ds^2 = -dt^2 + e^{2Ht} \frac{d\rho^2}{(1 + \frac{1}{4}k\rho^2)^2}.
\] (67)

5 Conclusion

In this paper we investigated the symmetric teleparallel gravity in two-dimensions. After giving the orthogonal, irreducible decomposition of the non-metricity tensor under the 2D Lorentz group, we wrote down the most general four-parameter Lagrangian density 2-form quadratic in the non-metricity tensor. We obtained the variational field equations and for any particular choice of coupling constants, we have shown that the corresponding field equations in a natural gauge admit both the static Schwarzschild solution and an inflationary cosmological solution.
References


