Physics of Jordan cells

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Outline

Jordan cells in non-hermitian quantum mechanics

Jordan cells in logarithmic conformal field theories

Jordan cells in the holographic $AdS_3/LCFT_2$ correspondence

Jordan cells in condensed matter applications

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Quantum mechanics

One of the postulates of quantum mechanics:

> Time-evolution of closed system described by
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> $$i\partial_t|\Psi\rangle = H|\Psi\rangle$$
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> with hermitian Hamiltonian H

Consequence of hermiticity: Eigenvalues are real

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Very useful concept with many applications in physics!

Non-hermitian quantum mechanics

Let us step back a bit

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Decay rate $\Gamma =$ non-hermitian contribution to Hamiltonian

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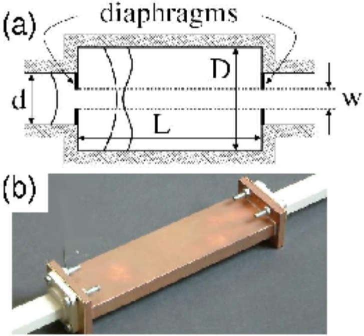
Surpisingly, the answer is yes

Experimental example

Example by Stefan Rotter et al. '04

Experimental setup:



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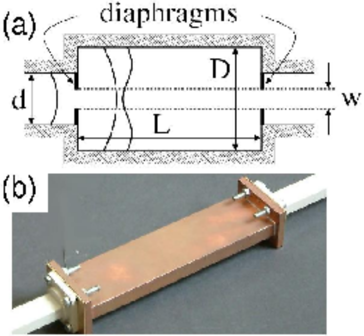
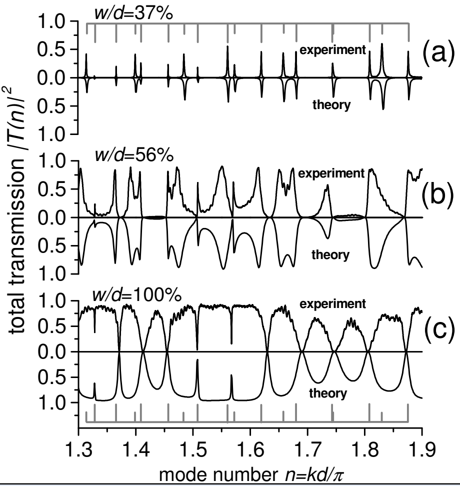


Total transmission probability:



Strong effect of non-hermiticity!

Critical points and Jordan cells

See "Non-Hermitian quantum mechanics" by Nimrod Moiseyev

Consider the Hamiltonian

$$H = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}$$

with Eigenvalues $E_\pm = \pm\sqrt{1+\lambda^2}$

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Non-hermitian critical points: $\lambda \to \pm i$
Eigenvector $c_\pm = (\pm i, 1)$ self-orthogonal: $c_\pm^2 = 0$

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Similarity trafo $J = A^{-1} H A$:

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Simplest example of Jordan cell in non-hermitian critical quantum mechanics!

Berry phase

Physical significance of critical points?

- Critical points and Jordan cells require infinite fine-tuning

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Physical significance of critical points: geometrical (Berry) phases!

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- Critical points experimentally accessible through geometric phases

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Conformal field theories

- CFT = Quantum field theory with invariance under translations, rotations + boosts, dilatations and special conformal transformations extends Poincare $SO(p,q)$ to $SO(p+1,q+1)$
  if $p=1$ and $q=d-1$: call this Lorentzian CFT$_d$
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- CFTs arise in physics in statistical mechanics, condensed matter physics, quantum field theory, string theory, gauge/gravity duality etc.
- Isometry group of (Lorentzian) AdS$_d$: $SO(2, d-1)$
  same as (Lorentzian) CFT$_{d-1}$
  Relevant for AdS$_d$/CFT$_{d-1}$ correspondence!

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- The 2- and 3-point correlators are fixed by conformal Ward identities.

$$\langle \mathcal{O}^R(\bar{z})\,\mathcal{O}^R(0)\rangle = \frac{c_R}{2\bar{z}^4}$$

$$\langle \mathcal{O}^L(z)\,\mathcal{O}^L(0)\rangle = \frac{c_L}{2z^4}$$

$$\langle \mathcal{O}^R(\bar{z})\,\mathcal{O}^R(\bar{z}')\,\mathcal{O}^R(0)\rangle = \frac{c_R}{\bar{z}^2\bar{z}'^2(\bar{z}-\bar{z}')^2}$$

$$\langle \mathcal{O}^L(z)\,\mathcal{O}^L(z')\,\mathcal{O}^L(0)\rangle = \frac{c_L}{z^2z'^2(z-z')^2}$$

Central charges $c_{L/R}$ determine key properties of CFT.

Correlators

The $c = 0$ catastrophe

Primary field $\mathcal{O}^M$ with conformal weights $(h, \bar{h})$:

$$\langle \mathcal{O}^M(z\,,\bar{z})\mathcal{O}^M(0\,,0)\rangle = \frac{A}{z^{2h}\,\bar{z}^{2\bar{h}}}$$

OPE:

$$\mathcal{O}^M(z\,,\bar{z})\mathcal{O}^M(0\,,0) \sim \frac{A}{z^{2h}\,\bar{z}^{2\bar{h}}}\Big(1 + \frac{2h}{c}\,z^2\mathcal{O}^L(0) + \dots\Big)$$

Problem: divergence for $c \to 0$

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Problem: divergence for $c \to 0$

Possible resolutions in limit $c \to 0$:

- weights vanish $(h, \bar{h}) \to (0, 0)$
- normalization vanishes $A \to 0$
- other operator(s) arise with $h \to 2$, which cancel divergence

Focus on last possibility

Correlators in logarithmic conformal field theories

Aghamohammadi, Khorrami & Rahimi Tabar '97; Kogan & Nichol '01; Rasmussen '04

Suppose now that primary has conformal weights $(2+\varepsilon\,,\varepsilon)$:

$$\langle\mathcal{O}^M(z,\bar{z})\,\mathcal{O}^M(0,0)\rangle = \frac{\hat{B}}{z^{4+2\varepsilon}\,\bar{z}^{2\varepsilon}}$$

Suppose that limits exist:

$$b_L := \lim_{c_L\to 0} -\frac{c_L}{\varepsilon} \neq 0 \qquad B := \lim_{c_L\to 0}\big(\hat{B} + \frac{2}{c_L}\big)$$

Define log operator:

$$\mathcal{O}^{\log} = b_L\,\frac{\mathcal{O}^L}{c_L} + \frac{b_L}{2}\,\mathcal{O}^M$$

Obtain 2-point correlators:

$$\langle\mathcal{O}^L(z)\mathcal{O}^L(0,0)\rangle = 0$$

$$\langle\mathcal{O}^L(z)\mathcal{O}^{\log}(0,0)\rangle = \frac{b_L}{2z^4}$$

$$\langle\mathcal{O}^{\log}(z,\bar{z})\mathcal{O}^{\log}(0,0)\rangle = -\frac{b_L\ln(m_L^2|z|^2)}{z^4}$$

Critical points and Jordan cells

If EMT acquires log partner Hamiltonian cannot be diagonalized

$$H \begin{pmatrix} \mathcal{O}^{\log} \\ \mathcal{O}^{L} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \mathcal{O}^{\log} \\ \mathcal{O}^{L} \end{pmatrix}$$

Consider only situations where J is diagonalizable:

$$J \begin{pmatrix} \mathcal{O}^{\log} \\ \mathcal{O}^{L} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \mathcal{O}^{\log} \\ \mathcal{O}^{L} \end{pmatrix}$$

Appearance of Jordan cell = defining feature of log CFTs

Note: Jordan cell can be higher rank than 2, but consider only rank 2 case here

LCFTs: Gurarie '93 Reviews on LCFTs: Flohr '01; Gaberdiel '01

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- Hamiltonian acquires Jordan cell structure

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Motivations for studying gravity in 3 dimensions

- Quantum gravity
  - Address conceptual issues of quantum gravity
  - Black hole evaporation, information loss, black hole microstate counting, virtual black hole production, ...
  - Technically much simpler than 4D or higher D gravity
  - Integrable models: powerful tools in physics (Coulomb problem, Hydrogen atom, harmonic oscillator, ...)
  - Models should be as simple as possible, but not simpler
- Gauge/gravity duality
  - Deeper understanding of black hole holography
  - $\text{AdS}_3/\text{CFT}_2$ correspondence best understood
  - Quantum gravity via AdS/CFT? (Witten '07, Li, Song, Strominger '08)
  - Applications to 2D condensed matter systems?
  - Gauge gravity duality beyond standard AdS/CFT: warped AdS, asymptotic Lifshitz, non-relativistic CFTs, logarithmic CFTs, ...
- Physics
  - Cosmic strings (Deser, Jackiw, 't Hooft '84, '92)
  - Black hole analog systems in condensed matter physics (graphene, BEC, fluids, ...)

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- Higher derivative theories of gravity can have massive graviton excitations and thus are locally non-trivial

Example: Topologically massive gravity (Deser, Jackiw & Templeton '82)

$$I_{\text{TMG}} = \frac{1}{16\pi\, G} \int \mathrm{d}^3x\sqrt{-g}\,\Big[R + \frac{2}{\ell^2} + \frac{1}{2\mu}\,\varepsilon^{\lambda\mu\nu}\,\Gamma^\rho{}_{\lambda\sigma}\big(\partial_\mu\Gamma^\sigma{}_{\nu\rho} + \frac{2}{3}\,\Gamma^\sigma{}_{\mu\tau}\Gamma^\tau{}_{\nu\rho}\big)\Big]$$

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Equations of motion:

$$R_{\mu\nu}-\frac{1}{2}\,g_{\mu\nu}R-\frac{1}{\ell^2}\,g_{\mu\nu}+\frac{1}{\mu}\,C_{\mu\nu}=0$$

with the Cotton tensor defined as

$$C_{\mu\nu}=\frac{1}{2}\,\varepsilon_\mu{}^{\alpha\beta}\nabla_\alpha R_{\beta\nu}+(\mu\leftrightarrow\nu)$$

Asymptotically AdS

Advantages of a negative cosmological constant in 3D gravity:

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- Do not have to discuss S-matrix
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Working definiton of asymptotically locally AdS_3:

$$g = \bar{g} + h \qquad \bar{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}y^2}{y^2}$$

with the state-dependent part h near the boundary $y = 0$:

$$\begin{pmatrix} h_{++} = o(1/y^2) & h_{+-} = \mathcal{O}(1) & h_{+y} = \mathcal{O}(1) \\ & h_{--} = o(1/y^2) & h_{-y} = \mathcal{O}(1) \\ & & h_{yy} = \mathcal{O}(1) \end{pmatrix}$$

Asymptotic symmetry group
Kinematics of asymptotically AdS_3

ASG = boundary preserving trafos / trivial gauge trafos

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Boundary preserving trafos

$$\mathcal{L}_\xi(\bar{g} + h) = \mathcal{O}(h)$$

generated by vector fields ξ with

$$\xi^+ = \varepsilon^+(x^+) + y^2 \dots$$
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asymptotic symmetry algebra generated by

$$\varepsilon^+(x^+)\partial_+=\sum_n L_n e^{inx^+}\qquad\varepsilon^-(x^-)\partial_-=\sum_n \bar{L}_n e^{inx^-}$$

Canonical realization of ASG

Dynamics of asymptotically AdS_3

Asymptotic symmetry algebra: two copies of Witt algebra (theory-independent!)

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Brown, Henneaux '86: canonical realization of ASG can lead to central extension (theory-dependent!)

$$[L_n\,, L_m] = (n-m)L_{n+m} + \frac{c}{12}\,(n^3-n)\,\delta_{n+m,0}$$

In Einstein gravity:

$$c = \bar{c} = \frac{3\ell}{2G}$$

Asymptotic symmetry algebra: two copies of Witt algebra (theory-independent!)

$$[L_n\,, L_m] = (n-m) L_{n+m} \qquad [\bar{L}_n\,, \bar{L}_m] = (n-m) \bar{L}_{n+m}$$

Brown, Henneaux '86: canonical realization of ASG can lead to central extension (theory-dependent!)

$$[L_n\,, L_m] = (n-m) L_{n+m} + \frac{c}{12}\,(n^3-n)\,\delta_{n+m,0}$$

In Einstein gravity:

$$c = \bar{c} = \frac{3\ell}{2G}$$

In topologically massive gravity (Kraus & Larsen '05):

$$c = \frac{3\ell}{2G}\,\Big(1 - \frac{1}{\mu\ell}\Big) \qquad \bar{c} = \frac{3\ell}{2G}\,\Big(1 + \frac{1}{\mu\ell}\Big)$$

Critical points in massive gravity

TMG at the chiral point is TMG with the tuning

$$\mu \, \ell = 1$$

between the cosmological constant and the Chern–Simons coupling.
Why special? (Li, Song & Strominger '08)

$$c = 0 \qquad \bar{c} = \frac{3\ell}{G}$$

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Interesting possibilities:

- Dual CFT could be chiral (Li, Song & Strominger '08)
- Dual CFT could be logarithmic (DG & Johansson '08)

Note: similar critical points exist in generic higher derivative gravity (DG, Johansson & Zojer '10)

Linearization around AdS background.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

Line-element $\bar{g}_{\mu\nu}$ of pure AdS:

$$\mathrm{d}\bar{s}^2_{\mathrm{AdS}} = \bar{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \ell^2\big(-\cosh^2\rho\,\mathrm{d}\tau^2 + \sinh^2\rho\,\mathrm{d}\phi^2 + \mathrm{d}\rho^2\big)$$

Isometry group: $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$
Useful to introduce light-cone coordinates $u = \tau + \phi$, $v = \tau - \phi$.
The $SL(2,\mathbb{R})_L$ generators

$$L_0 = i\partial_u$$

$$L_{\pm 1} = ie^{\pm iu}\Big[\frac{\cosh 2\rho}{\sinh 2\rho}\partial_u - \frac{1}{\sinh 2\rho}\partial_v \mp \frac{i}{2}\,\partial_\rho\Big]$$

obey the algebra $[L_0, L_{\pm 1}] = \mp L_{\pm 1}$, $[L_1, L_{-1}] = 2L_0$.
The $SL(2,\mathbb{R})_R$ generators $\bar{L}_0, \bar{L}_{\pm 1}$ obey same algebra, but with

$$u \leftrightarrow v\,, \qquad L \leftrightarrow \bar{L}$$

Linearization around AdS background.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

leads to linearized EOM that are third order PDE

$$G^{(1)}_{\mu\nu} + \frac{1}{\mu} C^{(1)}_{\mu\nu} = (\mathcal{D}^R \mathcal{D}^L \mathcal{D}^M h)_{\mu\nu} = 0 \tag{1}$$

with three mutually commuting first order operators

$$(\mathcal{D}^{L/R})_\mu{}^\nu = \delta^\nu_\mu \pm \ell\, \varepsilon_\mu{}^{\alpha\nu} \bar{\nabla}_\alpha \qquad (\mathcal{D}^M)_\mu{}^\nu = \delta^\nu_\mu + \frac{1}{\mu} \varepsilon_\mu{}^{\alpha\nu} \bar{\nabla}_\alpha$$

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Three linearly independent solutions to (1):

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At chiral point left (L) and massive (M) branches coincide!

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At chiral point: get log solution (DG & Johansson '08)

$$h^{\log}_{\mu\nu} = \lim_{\mu\ell\to 1}\frac{h^M_{\mu\nu}(\mu\ell) - h^L_{\mu\nu}}{\mu\ell - 1}$$

with property

$$\big(\mathcal{D}^L h^{\log}\big)_{\mu\nu} = \big(\mathcal{D}^M h^{\log}\big)_{\mu\nu} \neq 0\,, \qquad \big((\mathcal{D}^L)^2 h^{\log}\big)_{\mu\nu} = 0$$

Linearization around AdS background.

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Log mode exhibits interesting property:

$$H \begin{pmatrix} h^{\log} \\ h^L \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} h^{\log} \\ h^L \end{pmatrix}$$
$$J \begin{pmatrix} h^{\log} \\ h^L \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} h^{\log} \\ h^L \end{pmatrix}$$

Here $H = L_0 + \bar{L}_0 \sim \partial_t$ is the Hamilton operator and $J = L_0 - \bar{L}_0 \sim \partial_\phi$ the angular momentum operator.

Such a Jordan form of H and J is defining property of a logarithmic CFT!

Properties of logarithmic mode:

- Perturbative solution of linearized EOM, but not pure gauge

Checks of LCFT conjecture
Finiteness

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- Energy of logarithmic mode is finite

$$E^{\log} = -\frac{47}{1152 G\,\ell^3}$$

and negative $\to$ instability! (DG & Johansson '08)

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but violates Brown–Henneaux boundary conditions! ($\gamma_{ij}^{(1)}\big|_{\mathrm{BH}} = 0$)

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- Consistent log boundary conditions replacing Brown–Henneaux (DG & Johansson '08, Martinez, Henneaux & Troncoso '09)
- Brown–York stress tensor is finite and traceless, but not chiral
- Log mode persists non-perturbatively, as shown by Hamilton analysis (DG, Jackiw & Johansson '08, Carlip '08)

Checks of LCFT conjecture

Correlators

- Reminder: any CFT has conserved traceless EMT

$$T_{z\bar{z}} = 0 \qquad T_{zz} = \mathcal{O}^L(z) \qquad T_{\bar{z}\bar{z}} = \mathcal{O}^R(\bar{z})$$

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- 2- and 3-point correlators fixed by conformal Ward identities

$$\langle \mathcal{O}^R(\bar z)\, \mathcal{O}^R(0)\rangle = \frac{c_R}{2\bar z^4}$$

$$\langle \mathcal{O}^L(z)\, \mathcal{O}^L(0)\rangle = \frac{c_L}{2z^4}$$

$$\langle \mathcal{O}^L(z)\, \mathcal{O}^R(0)\rangle = 0$$

$$\langle \mathcal{O}^R(\bar z)\, \mathcal{O}^R(\bar z')\, \mathcal{O}^R(0)\rangle = \frac{c_R}{\bar z^2 \bar z'^2 (\bar z - \bar z')^2}$$

$$\langle \mathcal{O}^L(z)\, \mathcal{O}^L(z')\, \mathcal{O}^L(0)\rangle = \frac{c_L}{z^2 z'^2 (z - z')^2}$$

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Central charges $c_{L/R}$ determine key properties of CFT.

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- Suppose there is an additional operator $\mathcal{O}^M$ with conformal weights $h=2+\varepsilon$, $\bar h=\varepsilon$

$$\langle\mathcal{O}^M(z,\bar z)\,\mathcal{O}^M(0,0)\rangle=\frac{\hat B}{z^{4+2\varepsilon}\,\bar z^{2\varepsilon}}$$

which degenerates with $\mathcal{O}^{\mathrm{L}}$ in limit $c_L\propto\varepsilon\to 0$

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- Then energy momentum tensor acquires logarithmic partner $\mathcal{O}^{\log}$

$$\mathcal{O}^{\log} = b_L\, \frac{\mathcal{O}^L}{c_L} + \frac{b_L}{2}\, \mathcal{O}^M$$

where

$$b_L := \lim_{c_L \to 0} -\frac{c_L}{\varepsilon} \neq 0$$

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- Then energy momentum tensor acquires logarithmic partner $\mathcal{O}^{\log}$
- Some 2-point correlators:

$$\langle \mathcal{O}^L(z)\mathcal{O}^L(0,0)\rangle = 0$$

$$\langle \mathcal{O}^L(z)\mathcal{O}^{\log}(0,0)\rangle = \frac{b_L}{2z^4}$$

$$\langle \mathcal{O}^{\log}(z,\bar{z})\mathcal{O}^{\log}(0,0)\rangle = -\frac{b_L \ln\left(m_L^2|z|^2\right)}{z^4}$$

"New anomaly" b_L determines key properties of logarithmic CFT.

Checks of LCFT conjecture

Correlators

If LCFT conjecture is correct then following procedure must work:

- Calculate non-normalizable modes for left, right and logarithmic branches by solving linearized EOM on gravity side
- According to $\text{AdS}_3/\text{LCFT}_2$ dictionary these non-normalizable modes are sources for corresponding operators in the dual CFT
- Calculate 2- and 3-point correlators on the gravity side, e.g. by plugging non-normalizable modes into second and third variation of the on-shell action
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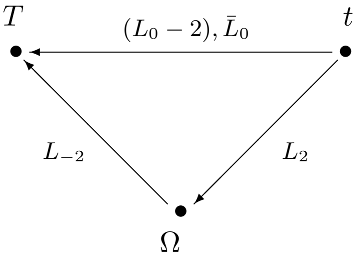
- Works at level of 2-point correlators (Skenderis, Taylor & van Rees '09, DG & Sachs '09)
- Works at level of 3-point correlators (DG & Sachs '09)
- Value of new anomaly: $b_L = -c_R = -3\ell/G$

Checks of LCFT conjecture

1-loop partition function (Gaberdiel, DG & Vassilevich '10)

Structure of low-lying states in LCFT:



Total partition function of Virasoro descendants

$$Z^0_{\text{LCFT}} = Z_\Omega + Z_{\text{t}} = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \Big(1 + \frac{q^2}{|1-q|^2}\Big)$$

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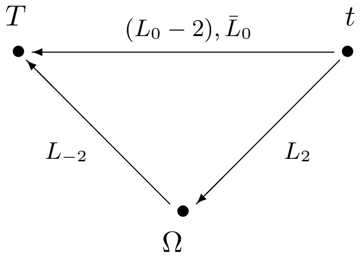

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Comparison with 1-loop calculation in Euclidean path integral approach to quantum gravity:

$$Z_{\text{TMG}} = \int_{\text{torus b.c.}} \mathcal{D}h_{\mu\nu} \times \text{ghost} \times \exp\big(-\delta^2 S(h)\big) = Z_{\text{Ein}} \times \det(\mathcal{D}^L)^{-1/2}$$

Calculating 1-loop determinant yields Einstein gravity result times another determinant.

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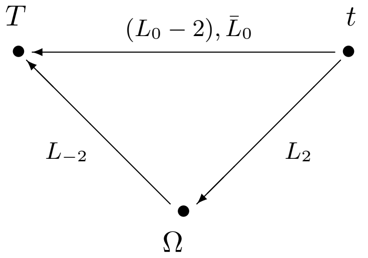

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$$\ln\det(\mathcal{D}^L)^{-1/2} = \sum_{n=1}^{\infty} \frac{1}{n} \, \frac{q^{2n}}{(1-q^n)(1-\bar{q}^n)}$$

Heat kernel methods allow to determine the new 1-loop determinant.

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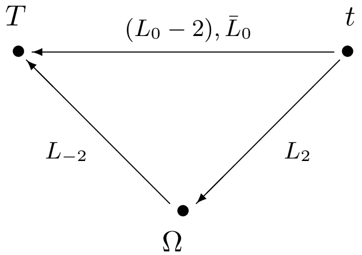

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$$Z_{\text{TMG}} = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \prod_{m=2}^{\infty} \prod_{\bar{m}=0}^{\infty} \frac{1}{1-q^m \bar{q}^{\bar{m}}}$$

The final result consists of two factors, an Einstein piece and a new contribution from the log modes.

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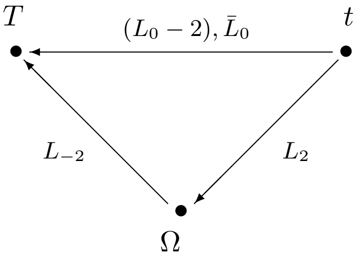

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$$Z_{\text{TMG}} = Z^0_{\text{LCFT}} + \sum_{h,\bar{h}} N_{h,\bar{h}}\, q^h \bar{q}^{\bar{h}} \prod_{n=1}^{\infty} \frac{1}{|1-q^n|^2}$$

All multiplicity coefficients $N_{h,\bar{h}}$ can be shown to be non-negative.
Fairly non-trivial test of the LCFT conjecture!

Generalizations

Log CFTs have taught us a great deal about critical TMG!

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Generalizations beyond topologically massive gravity at the critical point?

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Generalizations beyond topologically massive gravity at the critical point?

- 3D: generic massive gravity theories (DG, Johansson, Zojer '10)
  - New massive gravity, generalized massive gravity, higher curvature gravity, ...
  - higher rank Jordan cells possible in some of these models
  - qualitatively new log CFTs arise: no log partner of energy-momentum tensor
  - beyond the SUGRA approximation?

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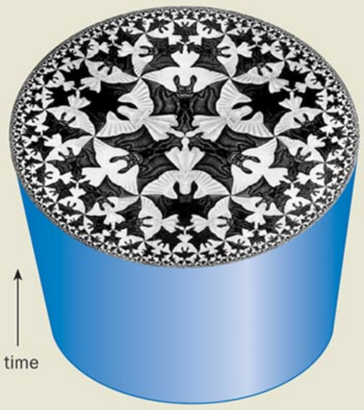
- Higher spin log gravity? (Chen, Long, Wu; Bagchi, Lal, Saha, Sahoo '11)

Holographic conclusions

- 3-dimensional gravity duals for 2-dimensional log CFTs seem to exist



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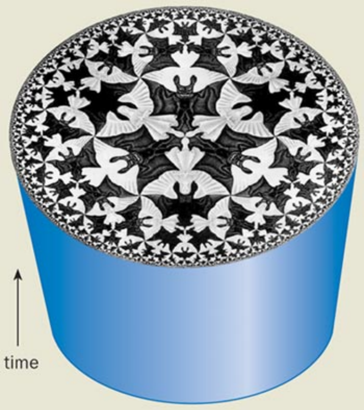

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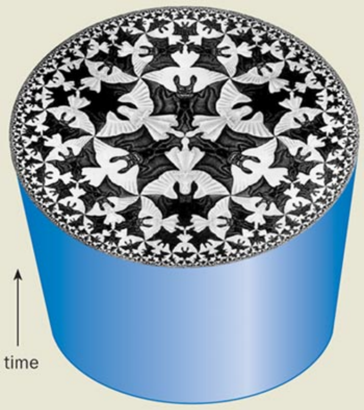

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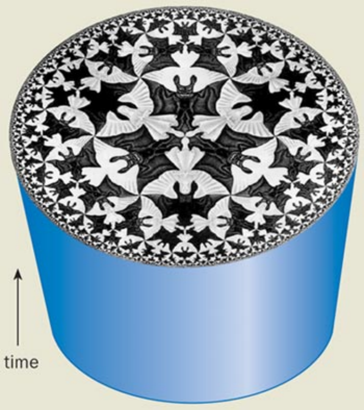

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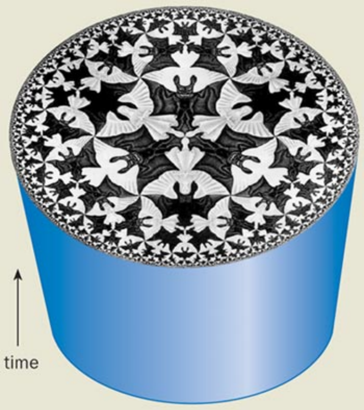
- intriguing perspective: higher dimensional log CFTs?



Outline

Jordan cells in non-hermitian quantum mechanics

Jordan cells in logarithmic conformal field theories

Jordan cells in the holographic $AdS_3/LCFT_2$ correspondence

Jordan cells in condensed matter applications

Systems with quenched disorder

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- Idea: Apply $\mathrm{AdS}_3/\mathrm{LCFT}_2$ to describe strongly coupled LCFTs!

Some literature on condensed matter applications of LCFTs

- Cardy '99 Logarithmic correlations in Quenched Random Magnets and Polymers
- Gurarie & Ludwig '99 Conformal algebras of 2D disordered systems
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More applications awaiting to be discovered!