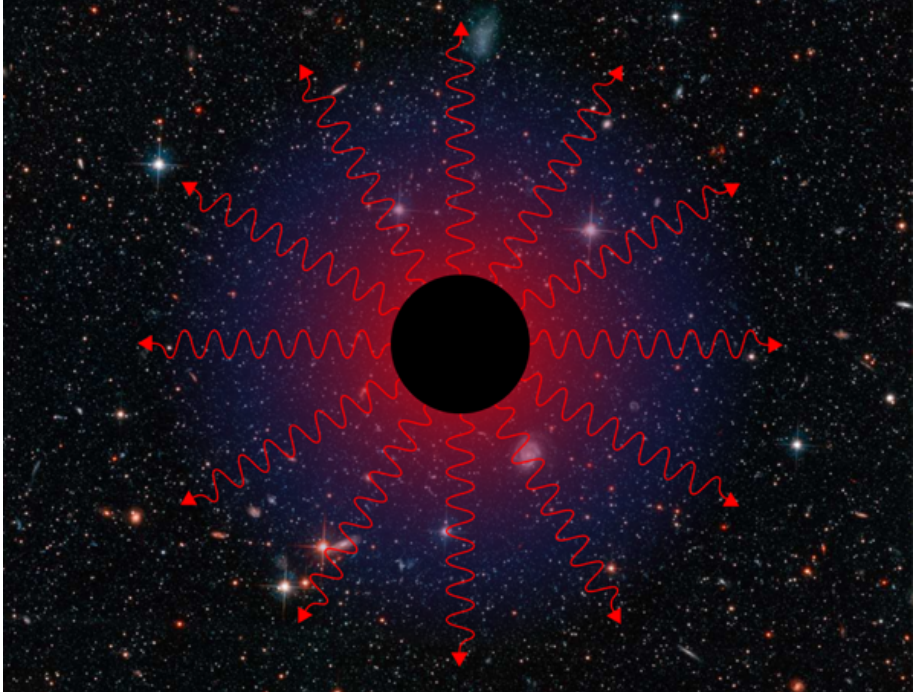# Soft Heisenberg Hair

Daniel Grumiller

Institute for Theoretical Physics
TU Wien

Imperial College London, Δ Seminar Talk, March 2019

FWF

Two simple punchlines

1. Heisenberg algebra

$$[X_n,\,P_m] = i\,\delta_{n,\,m}$$

fundamental not only in quantum mechanics
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at least in three spacetime dimensions

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2. Black hole microstates identified as specific "soft hair" descendants
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based on work (2016-2019) with

- Hamid Afshar, Shahin Sheikh-Jabbari, Zahra Mirzaiyan [IPM Teheran]
- Martin Ammon [U. Jena]
- Stephane Detournay, Wout Merbis, Stefan Prohazka, Max Riegler [ULB]
- Hernán González [AIU Santiago]
- Philip Hacker, Raphaela Wutte, Céline Zwikel [TU Wien]
- Alfredo Perez, David Tempo, Ricardo Troncoso [CECs Valdivia]
- Hossein Yavartanoo [ITP Beijing]

Outline

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Boundary charges

Science is a differential equation. Religion is a boundary condition. — Alan Turing

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> All boundary condition preserving gauge transformations
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- ▶ Choice of boundary conditions determines asymptotic symmetries

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Daniel Grumiller — Soft Heisenberg Hair   Boundary charges   6/31

Asymptotic symmetries in gravity

▶ Impose some bc's at (asymptotic or actual) boundary:

$$\lim_{r \to r_b} g_{\mu\nu}(r,\, x^i) = \bar{g}_{\mu\nu}(r_b,\, x^i) + \delta g_{\mu\nu}(r_b,\, x^i)$$

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$\xi^\mu_{(0)}(r_b,\, x^i)$: generates asymptotic symmetries

subleading terms: generate trivial diffeos

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> **Definition of asymptotic symmetry algebra**
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> Lie bracket quotient algebra of asymptotic Killing vectors modulo trivial diffeos

Simple example (based on unpublished notes with Salzer)

Asymptotic Rindler$_2$ spacetimes (in Eddington–Finkelstein gauge)

▶ Consider class of 2d metrics, partially gauge-fixed

$$g_{rr}(r,\, u) = 0$$
$$g_{ur}(r,\, u) = -1$$
$$g_{uu}(r,\, u) = \delta g(u)r + \mathcal{O}(1)$$

expanded for large r

Note: Ricci scalar tends to zero for large r

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▶ asymptotic symmetry algebra ("BMS$_2$"):

$$\big[\xi(\epsilon_1,\, \eta_1),\, \xi(\epsilon_2,\, \eta_2)\big]_{\mathrm{Lie}} = \xi\big(\epsilon_1\epsilon_2' - \epsilon_2\epsilon_1',\, (\epsilon_1\eta_2 - \epsilon_2\eta_1)'\big)$$

Lie bracket algebra of asymptotic Killing vectors is infinite dimensional here

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▶ in Fourier-modes $L_n := \xi(\epsilon = ie^{inu},\,\eta = 0)$, $J_n := \xi(\epsilon = 0,\,\eta = ie^{inu})$:

$$[L_n,\,L_m]_{\text{Lie}} = (n-m)\,L_{n+m} \qquad [J_n,\,J_m]_{\text{Lie}} = 0 \qquad [L_n,\,J_m]_{\text{Lie}} = -(n+m)\,J_{n+m}$$

Witt algebra (spin-2) with current-type algebra (spin-0)

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Asymptotic Rindler$_2$ spacetimes (asymptotically in Eddington–Finkelstein gauge)

▶ Consider class of 2d metrics

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▶ dropping partial gauge-fixing does not change asymptotic symmetries
  instead, switches on trivial diffeos

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time-independent Schrödinger equation:

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) = E\psi(x)$$

look for (normalizable) bound state solutions, $E < 0$
  ► Dirichlet bc's: no bound states
  ► Neumann bc's: no bound states

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  ▶ Robin bc's

$$\left(\psi + \alpha\psi'\right)\big|_{x=0^+} = 0 \qquad \alpha \in \mathbb{R}^+$$

lead to one bound state

$$\psi(x)\big|_{x \geq 0} = \sqrt{\frac{2}{\alpha}}\, e^{-x/\alpha}$$

with energy $E = -1/\alpha^2$, localized exponentially near $x = 0$

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▶ changing boundary conditions can change physical spectrum
▶ to distinguish asymptotic symmetries from trivial gauge trafos: either
  use Noether's second theorem and covariant phase space analysis or
  perform Hamiltonian analysis in presence of boundaries

Some references:
  ▶ covariant phase space: Lee, Wald '90, Iyer, Wald '94 and Barnich,
    Brandt '02
  ▶ review: see Compère, Fiorucci '18 and refs. therein
  ▶ canonical analysis: Arnowitt, Deser, Misner '59, Regge, Teitelboim '74
    and Brown, Henneaux '86
  ▶ review: see Bañados, Reyes '16 and refs. therein

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- ▶ in Hamiltonian language: gauge generator $G[\epsilon]$ varies as

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Trivial gauge transformations generated by some ϵ with $Q[\epsilon] = 0$

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Z: possible central extension of asymptotic symmetry algebra

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> **Canonical realization of asymptotic symmetries**
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> Poisson (or Dirac) bracket algebra of canonical boundary charges

▶ physical phase space falls into representations of asymptotic symmetry algebra $\Rightarrow$ useful e.g. for holography

Simple example: abelian Chern–Simons

- abelian Chern–Simons action (on cylinder)

$$I[A] = \frac{k}{4\pi} \int_{\mathbb{R} \times \Sigma} A \wedge \mathrm{d}A$$

Note: topological QFT with no local physical degrees of freedom

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$$Q[\epsilon] = \frac{k}{2\pi} \oint_{\partial \Sigma} \epsilon \, A$$

▶ choice of bc's

$$\lim_{r \to \infty} A = \mathcal{J}(\varphi) \, \mathrm{d}\varphi + \mu \, \mathrm{d}t$$

preserved by $\epsilon = \eta(\varphi) +$ subleading

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▶ Fourier modes $J_n \sim \oint \mathcal{J} e^{in\varphi}$ yield $u(1)_k$ current algebra, $i\{J_n,\, J_m\} = \frac{k}{2}\, n\, \delta_{n+m,\,0}$

Edge states

▶ changing boundary charges changes physical state

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see e.g. Halperin '82, Witten '89, or Balachandran, Chandar, Momen '94

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    - ▶ descendants of vacuum are examples of edge states

    $$|\text{edge}(\{n_i\})\rangle = \prod_{\{n_i > 0\}} J_{-n_i}|0\rangle$$

    e.g.

    $$|\text{edge}(\{1, 1, 42\})\rangle = J_{-1}^2 J_{-42}|0\rangle$$

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- ▶ theories with no local physical degrees of freedom can have edge states! ⇒ perhaps cleanest example of holography

Outline

Old idea by Strominger '97 and Carlip '98

> **Main idea**
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> Impose existence of non-extremal horizon
> as boundary condition on state space

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Main idea

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▶ Want to understand Bekenstein–Hawking entropy

$$S_{\mathrm{BH}} = \frac{A}{4G} + \mathcal{O}(\ln(A/G))$$

Motivation for near horizon boundary conditions
Old idea by Strominger '97 and Carlip '98

> **Main idea**
>
> Impose existence of non-extremal horizon
> as boundary condition on state space

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Explicit form of near horizon boundary conditions

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1. Rindler approximation

$$ds^2 = -\kappa^2 r^2 \, dt^2 + dr^2 + \Omega_{ab}(t, \, x^c) \, dx^a \, dx^b + \dots$$

$r \to 0$: Rindler horizon

κ: surface gravity

Ω_{ab}: metric transversal to horizon

$\dots$: terms of higher order in r or rotation terms

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2. Surface gravity is state-independent

$$\delta\kappa = 0$$

Explicit form of near horizon boundary conditions
See Donnay, Giribet, Gonzalez, Pino '15 and Afshar et al '16

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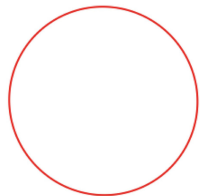
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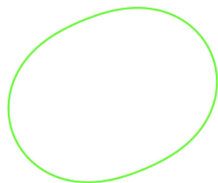
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4. Remaining terms fixed by consistency of canonical boundary charges

Black holes can be deformed into black flowers Afshar et al. 16
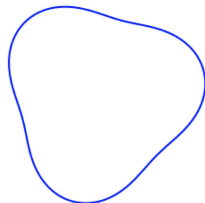
Horizon can get excited by area preserving shear-deformations

$k = 1$ $k = 2$ $k = 3$
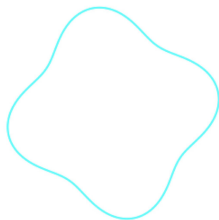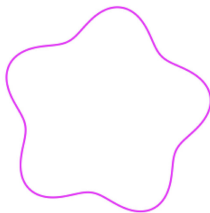
$k = 4$ $k = 5$ $k = 6$

Near horizon symmetries = "asymptotic symmetries" for near horizon bc's
Restrict for the time being to AdS$_3$ black holes (BTZ)

Simplification in 3d:

$$\mathrm{d}s^2 = \Big[-\kappa^2 r^2\ \mathrm{d}t^2 + \mathrm{d}r^2 + \gamma^2(\varphi)\ \mathrm{d}\varphi^2 + 2\kappa\,\omega(\varphi)\,r^2\ \mathrm{d}t\,\mathrm{d}\varphi \Big]\left(1 + \mathcal{O}(r^2)\right)$$

▶ Map from round S^1 to Fourier-excited S^1: diffeo $\gamma(\varphi)\ \mathrm{d}\varphi = \mathrm{d}\tilde\varphi$

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▶ Trivial or non-trivial?
  Answer provided by boundary charges!
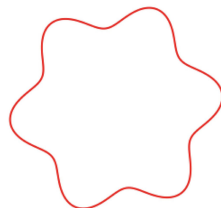
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▶ Map from round S^1 to Fourier-excited S^1: diffeo $\gamma(\varphi) \, d\varphi = d\tilde\varphi$
▶ Non-trivial diffeo!
▶ Canonical analysis yields

$$Q^\pm[\epsilon^\pm] \sim \oint d\varphi\,\epsilon^\pm(\varphi)\left(\gamma(\varphi) \pm \omega(\varphi)\right)$$

where $\epsilon^\pm$ are functions appearing in asymptotic Killing vectors

charge conservation follows from on-shell relations $\partial_t\gamma = 0 = \partial_t\omega$

explains last word in title: γ and ω are hair of black hole

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$$[\mathcal{J}_n^{\pm}, \, \mathcal{J}_m^{\pm}] = \tfrac{1}{2} \, n \, \delta_{n+m,\,0}$$

Two $u(1)$ current algebras! Afshar et al. 16

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$$[\mathcal{J}_n^\pm,\, \mathcal{J}_m^\pm] = \tfrac{1}{2}\, n\, \delta_{n+m,\, 0}$$

▶ Isomorphic to Heisenberg algebras plus center

$$[X_n,\, P_m] = i\, \delta_{n,m} \qquad\qquad [P_0,\, X_n] = 0 = [X_0,\, P_n]$$

$$P_0 = \mathcal{J}_0^+ + \mathcal{J}_0^-,\, X_n = \mathcal{J}_n^+ - \mathcal{J}_{-n}^-,\, P_n = 2i/n(\mathcal{J}_{-n}^+ + \mathcal{J}_n^-) \text{ for } n \neq 0$$

Unique features of near horizon boundary conditions

1. All states allowed by bc's have same temperature

   By contrast: asymptotically AdS or flat space bc's allow for black hole states at different masses and hence different temperatures

Unique features of near horizon boundary conditions

1. All states allowed by bc's have same temperature
2. All states allowed by bc's are regular

   (in particular, they have no conical singularities at the horizon in the Euclidean formulation)

   By contrast: for given temperature not all states in theories with asymptotically AdS or flat space bc's are free from conical singularities; usually a unique black hole state is picked

Unique features of near horizon boundary conditions

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2. All states allowed by bc's are regular

   (in particular, they have no conical singularities at the horizon in the Euclidean formulation)

3. There is a non-trivial reducibility parameter ($=$ Killing vector)

   By contrast: for any other known (non-trivial) bc's there is no vector field that is Killing for all geometries allowed by bc's

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4. Technical feature: in Chern–Simons formulation of 3d gravity simple expressions in diagonal gauge

$$A^{\pm} = b^{\mp 1} \left(\mathrm{d} + a^{\pm} \right) b^{\pm 1}$$

$$a^{\pm} = L_0 \left(\left(\gamma(\varphi) \pm \omega(\varphi) \right) \mathrm{d}\varphi + \kappa \, \mathrm{d}t \right)$$

$$b = \exp \left[\left(L_+ - L_- \right) r/2 \right]$$

$L_{\pm}$ are $sl(2, \mathbb{R})$ raising/lowering generators
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5. Leads to soft Heisenberg hair (see next slides!)

Outline

Soft Heisenberg hair for BTZ

▶ Black flower excitations = hair of black holes
Algebraically, excitations from descendants

$$|\text{black flower}\rangle \sim \prod_{n_i^\pm > 0} \mathcal{J}^+_{-n_i^+} \mathcal{J}^-_{-n_i^-} |\text{black hole}\rangle$$

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▶ Near horizon Hamiltonian = boundary charge associated with unit time-translations*

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commutes with all generators $\mathcal{J}^\pm_n$

* units defined by specifying κ

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with Hawking–Unruh-temperature

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δ refers to any variation of phase space variables allowed by the boundary conditions

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> Can we understand entropy law microscopically?

Given our soft Heisenberg hair, attack now entropy questions

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Regarding 1. and 3.: may expect decoupling of scales so that description of microstates does not need info about UV completion, but rather only some semi-classical "Bohr-like" input

Evidence for this: universality of BH entropy for large black holes

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▶ possible resolution: do not consider asymptotic but near horizon observer (i.e., employ near horizon bc's and symmetry algebra)

Fluff proposal (with Afshar, Sheikh-Jabbari '16 and also with Yavartanoo '17)
Semi-classical BTZ black hole microstates as near horizon descendants of vacuum

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subject to spectral constraint depending on black hole mass M and angular momentum J (measured by asymptotic observer)

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$$\sum_i n_i^{\pm} = \frac{c}{2} \left(M \pm J \right)$$

derived from Bohr-type quantization conditions
- ▶ quantization of central charge $c = 3/(2G)$ in integers
- ▶ quantization of conical deficit angles in integers over c
- ▶ black hole/particle correspondence
  (black hole = gas of coherent states of particles on AdS_3)

Microstates for BTZ black hole with mass M and angular momentum J:

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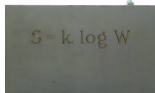Microstates for BTZ black hole with mass M and angular momentum J:

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$$p(N)\big|_{N\gg 1} \sim \frac{1}{4N\sqrt{3}} \exp\left(2\pi\sqrt{N/6}\right)$$

▶ to get entropy use Boltzmann's formula

$$S = \ln p\!\left(\tfrac{c}{2}\left(M+J\right)\right) + \ln p\!\left(\tfrac{c}{2}\left(M-J\right)\right)$$

(we set $k=1$ and $W=p$)

Check of fluff proposal

Microstates for BTZ black hole with mass M and angular momentum J:

$$|\mathcal{B}(\{n_i^\pm\})\rangle = \prod_{\{n_i^\pm > 0\}} \left(\mathcal{J}_{-n_i^+}^+ \cdot \mathcal{J}_{-n_i^-}^- \right) |0\rangle \qquad \sum_i n_i^\pm = \tfrac{c}{2}\,(M \pm J)$$

▶ count number of BTZ black hole microstates
▶ combinatorial problem: how many ways to decompose large positive integer $\tfrac{c}{2}\,(M \pm J)$ into sum of positive integers
▶ solved by Hardy, Ramanujan 1918

$$p(N)\big|_{N \gg 1} \sim \frac{1}{4N\sqrt{3}}\,\exp\left(2\pi\,\sqrt{N/6} \right)$$

▶ to get entropy use Boltzmann's formula

$$S = \ln p\big(\tfrac{c}{2}\,(M + J)\big) + \ln p\big(\tfrac{c}{2}\,(M - J)\big)$$

▶ leading order yields Cardy formula and hence the BH entropy

$$S = 2\pi\sqrt{\tfrac{c}{6}\,(M + J)} + 2\pi\sqrt{\tfrac{c}{6}\,(M - J)} = 2\pi\,P_0 = \frac{A}{4G} + \dots$$

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▶ leading + subleading order yields BH entropy plus log corrections

$$S = \frac{A}{4G} - 2\ln\left(A/(4G)\right) + \dots$$

Outline

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  works in any dimension, for any local geometry, for any reasonable
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* theories checked so far:
Einstein gravity with negative cosmological constant ($d \geq 3$)
Einstein gravity with vanishing cosmological constant ($d \geq 3$)
higher spin gravity ($d = 3$, principal embedding of $sl(2)$)
various massive gravity theories ($d = 3$)

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[*] for instance, for Schwarzschild

$$\{Q_{lm},\, P_{l'm'}\} = \frac{1}{8\pi G}\, \delta_{ll'}\, \delta_{mm'} \qquad l > 0 \qquad \{P_{00},\, \bullet\} = 0$$

Q_{lm}: spherical harmonics of area preserving shear deformations
P_{lm}: spherical harmonics of near horizon supertranslations
Entropy given by $S = 2\pi\, P_{00}$

Kerr has additional generators: area preserving twist deformations

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  might work more generally, but so far only checked BTZ black hole; needed Bohr-type rules to succeed

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Take-away messages:

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$$|\mathcal{B}(\{n_i^{\pm}\})\rangle = \prod_{\{n_i^{\pm} > 0\}} \left(\mathcal{J}_{-n_i^+}^+ \cdot \mathcal{J}_{-n_i^-}^- \right) |0\rangle \qquad \sum_i n_i^{\pm} = \text{fixed by } M, J$$

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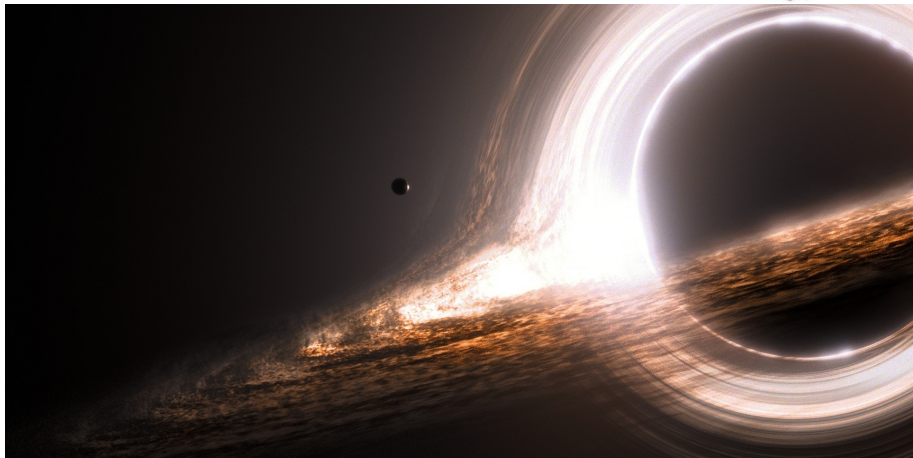
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► Microstate construction for non-extremal Kerr?

Thanks for your attention!

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with

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▶ thus, Lie-bracket replaced by modified Lie-bracket

$$[\xi_1,\, \xi_2]_{\text{mod}} = [\xi_1,\, \xi_2]_{\text{Lie}} + \delta_{\xi_2}\xi_1 - \delta_{\xi_1}\xi_2$$

main difference to DGGP, where ξ is state-independent!

Map to asymptotic variables

▶ Usual asymptotic AdS$_3$ connection with chemical potential μ:

$$\hat{A} = \hat{b}^{-1}(\,\mathrm{d}+\hat{\mathfrak{a}})\hat{b} \qquad \hat{\mathfrak{a}}_{\varphi} = L_+ - \tfrac{1}{2}\,\mathcal{L}\,L_-$$

$$\hat{b} = e^{\rho L_0} \quad \hat{\mathfrak{a}}_t = \mu L_+ - \mu' L_0 + \left(\tfrac{1}{2}\,\mu'' - \tfrac{1}{2}\,\mathcal{L}\mu\right) L_-$$

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$$\mathcal{L} = \tfrac{1}{2}\mathcal{J}^2 + \mathcal{J}'$$

Bonus slide II
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▶ Virasoro w. Brown–Henneaux central charge $\delta\mathcal{L} = 2\mathcal{L}\varepsilon' + \mathcal{L}'\varepsilon - \varepsilon'''$

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   $$\mathcal{J}_{cn} \sim \mathcal{W}_n^0$$

   Note non-local relation

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Note twisted periodicity conditions

$$\mathcal{W}^\nu(\varphi + 2\pi) = e^{-2\pi\nu i}\,\mathcal{W}^\nu(\varphi)$$

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   Maldacena, Maoz '00; Lunin, Maldacena, Maoz '02

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Bonus slide III
Some fluffy details

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   Justification 2: gives nice result