Near horizon dynamics of three dimensional black holes

Daniel Grumiller

Institute for Theoretical Physics
TU Wien

Seminar talk at ICTS, Bangalore, August, 2019

work with Wout Merbis, 1906.10694
Outline

Overture

Hamiltonian reduction

Near horizon boundary conditions

Near horizon Hamiltonian

KdV deformation

Conclusions
Outline

Overture

Hamiltonian reduction

Near horizon boundary conditions

Near horizon Hamiltonian

KdV deformation

Conclusions
Main message

Near horizon boundary action for 3-dimensional black holes

\[ S_{NH}[\Phi^+, \Phi^-] = \int dt d\sigma (\Pi^+ \dot{\Phi}^+ + \Pi^- \dot{\Phi}^- - \mathcal{H}_{NH}(\Phi^+, \Phi^-)) \]
Main message

- Near horizon boundary action for 3-dimensional black holes

\[ S_{\text{NH}}[\Phi^+, \Phi^-] = \int dt \, d\sigma (\Pi^+ \dot{\Phi}^+ + \Pi^- \dot{\Phi}^- - \mathcal{H}_{\text{NH}}(\Phi^+, \Phi^-)) \]

- Scalar fields \( \Phi^\pm \) denote left/right movers along the horizon

Purpose of talk: explain and derive results summarized above
Main message

- Near horizon boundary action for 3-dimensional black holes

\[ S_{\text{NH}}[\Phi^+, \Phi^-] = \int dt \, d\sigma \left( \Pi^+ \dot{\Phi}^+ + \Pi^- \dot{\Phi}^- - \mathcal{H}_{\text{NH}}(\Phi^+, \Phi^-) \right) \]

- Scalar fields \( \Phi^\pm \) denote left/right movers along the horizon
to reduce clutter: drop \( \pm \) decorations in rest of talk
Main message

- Near horizon boundary action for 3-dimensional black holes

\[ S_{\text{NH}}[\Phi^+, \Phi^-] = \int dt \, d\sigma (\Pi^+ \dot{\Phi}^+ + \Pi^- \dot{\Phi}^- - \mathcal{H}_{\text{NH}}(\Phi^+, \Phi^-)) \]

- Scalar fields \( \Phi^\pm \) denote left/right movers along the horizon
- Scalar fields are self-dual (Floreanini–Jackiw-like)

\[ \Pi \sim \Phi' \]
Main message

- Near horizon boundary action for 3-dimensional black holes

\[ S_{NH}[\Phi^+, \Phi^-] = \int dt \, d\sigma (\Pi^+ \dot{\Phi}^+ + \Pi^- \dot{\Phi}^- - \mathcal{H}_{NH}(\Phi^+, \Phi^-)) \]

- Scalar fields \( \Phi^\pm \) denote left/right movers along the horizon
- Scalar fields are self-dual (Floreanini–Jackiw-like)

\[ \Pi \sim \Phi' \]

- Near horizon Hamilton density is total derivative

\[ \mathcal{H}_{NH}(\Phi) \sim \zeta \Phi' \]

Manifestation of “softness” of near horizon excitations
Main message

- Near horizon boundary action for 3-dimensional black holes

\[ S_{\text{NH}}[\Phi^+, \Phi^-] = \int dt \, d\sigma \left( \Pi^+ \dot{\Phi}^+ + \Pi^- \dot{\Phi}^- - \mathcal{H}_{\text{NH}}(\Phi^+, \Phi^-) \right) \]

- Scalar fields \( \Phi^\pm \) denote left/right movers along the horizon

- Scalar fields are self-dual (Floreanini–Jackiw-like)

\[ \Pi \sim \Phi' \]

- Near horizon Hamilton density is total derivative

\[ \mathcal{H}_{\text{NH}}(\Phi) \sim \zeta \Phi' \]

Manifestation of “softness” of near horizon excitations

Purpose of talk: explain and derive results summarized above
Outline

Overture

Hamiltonian reduction

Near horizon boundary conditions

Near horizon Hamiltonian

KdV deformation

Conclusions
Einstein gravity in three dimensions as Chern–Simons theory

Einstein gravity in three dimensions useful toy model:

\[ I_{EH3}[g] = \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x \sqrt{-g} \left( R + \frac{2}{\ell^2} \right) + \hat{I}_{\partial \mathcal{M}} \]

- no local physical degrees of freedom \( \Rightarrow \) simple!
Einstein gravity in three dimensions as Chern–Simons theory

Einstein gravity in three dimensions useful toy model:

\[ I_{EH3}[g] = \frac{1}{16\pi G} \int_M d^3x \sqrt{-g} \left( R + \frac{2}{\ell^2} \right) + \hat{I}_{\partial M} \]

\[ \begin{align*}
\text{no local physical degrees of freedom} & \Rightarrow \text{simple!} \\
\text{rotating (BTZ) black hole solutions analogous to Kerr} \\
\end{align*} \]

\[ ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} \ dt^2 + \frac{\ell^2 r^2 \ dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2 \left( d\varphi - \frac{r_+ + r_-}{\ell r^2} \ dt \right)^2 \]
Einstein gravity in three dimensions as Chern–Simons theory

Einstein gravity in three dimensions useful toy model:

\[ I_{EH3}[g] = \frac{1}{16\pi G} \int_M d^3x \sqrt{-g} \left( R + \frac{2}{\ell^2} \right) + \hat{I}_{\partial M} \]

- no local physical degrees of freedom ⇒ simple!
- rotating (BTZ) black hole solutions analogous to Kerr

\[ ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} \, dt^2 + \frac{\ell^2 r^2 \, dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2 \left( d\varphi - \frac{r_+ + r_-}{\ell r^2} \, dt \right)^2 \]

- Brown–Henneaux asymptotic symmetries: 2 Virasoros (AdS$_3$/CFT$_2$)

\[ [L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0} \]

\[ c = \frac{3\ell}{2G} \]
Einstein gravity in three dimensions as Chern–Simons theory

Einstein gravity in three dimensions useful toy model:

\[ I_{\text{EH3}}[g] = \frac{1}{16\pi G} \int_M d^3x \sqrt{-g} \left( R + \frac{2}{\ell^2} \right) + \hat{I}_\partial \mathcal{M} \]

- no local physical degrees of freedom \( \Rightarrow \) simple!
- rotating (BTZ) black hole solutions analogous to Kerr

\[
ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2r^2} \ dt^2 + \frac{\ell^2 r^2 \ dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2 \left( d\varphi - \frac{r_+ + r_-}{\ell r^2} \ dt \right)^2
\]

- Brown–Henneaux asymptotic symmetries: 2 Virasoros \((\text{AdS}_3/\text{CFT}_2)\)

\[
[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0}
\]

\[ c = \frac{3\ell}{2G} = 6k \]

- Gauge theoretic formulation as Chern–Simons theory \([k = \ell/(4G)]\)

\[
I_{\text{CS}}[A] = \frac{k}{4\pi} \int_\mathcal{M} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + I_{\partial \mathcal{M}}
\]

\( SO(2, 2) \) connection \( A \) usually split into two \( \text{SL}(2, \mathbb{R}) \) connections; drop all \( \pm \) decorations & work with single sector
Hamiltonian analysis of Chern–Simons theory

- Hamiltonian action of Chern–Simons theory on cylinder adapted coordinates: $r$: radius, $\sigma \sim \sigma + 2\pi$: angle, $t$: time

$$I_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr} \left( A_r \dot{A}_\sigma - A_\sigma \dot{A}_r + 2A_t F_{\sigma r} \right) + I_{\partial \mathcal{M}}$$
Hamiltonian analysis of Chern–Simons theory

- Hamiltonian action of Chern–Simons theory on cylinder adapted coordinates: \( r \): radius, \( \sigma \sim \sigma + 2\pi \): angle, \( t \): time

\[
I_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr} \left( A_r \dot{A}_\sigma - A_\sigma \dot{A}_r + 2A_t F_{\sigma r} \right) + I_{\partial \mathcal{M}}
\]

- Constraint \( F_{\sigma r} = 0 \) locally solved by

\[
A_i = G^{-1} \partial_i G \quad G \in \text{SL}(2, \mathbb{R})
\]
Hamiltonian analysis of Chern–Simons theory

- Hamiltonian action of Chern–Simons theory on cylinder adapted coordinates:  $r$: radius, $\sigma \sim \sigma + 2\pi$: angle, $t$: time

$$I_{\text{CS}}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr} \left( A_r \dot{A}_\sigma - A_\sigma \dot{A}_r + 2A_t F_{\sigma r} \right) + I_{\partial \mathcal{M}}$$

- Constraint $F_{\sigma r} = 0$ locally solved by

$$A_i = G^{-1} \partial_i G \quad G \in \text{SL}(2, \mathbb{R})$$

- Gauge $\partial_\sigma A_r = A'_r = 0$ implies $G = g(t, \sigma)b(t, r)$

$$A_\sigma = b^{-1} a_\sigma b \quad a_\sigma = g^{-1} g' \quad A_r = b^{-1} \partial_r b$$
Hamiltonian analysis of Chern–Simons theory

- Hamiltonian action of Chern–Simons theory on cylinder adapted coordinates: \( r \): radius, \( \sigma \sim \sigma + 2\pi \): angle, \( t \): time

\[
I_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr} \left( A_r \dot{A}_\sigma - A_\sigma \dot{A}_r + 2A_t F_{\sigma r} \right) + I_{\partial \mathcal{M}}
\]

- constraint \( F_{\sigma r} = 0 \) locally solved by

\[
A_i = G^{-1} \partial_i G \quad G \in \text{SL}(2, \mathbb{R})
\]

- gauge \( \partial_\sigma A_r = A'_r = 0 \) implies \( G = g(t, \sigma)b(t, r) \)

\[
A_\sigma = b^{-1} a_\sigma b \quad a_\sigma = g^{-1} g' \quad A_r = b^{-1} \partial_r b
\]

- for formulating boundary conditions related convenient Ansatz:

\[
A(t, \sigma, r) = b^{-1}(r) \left( d + a(t, \sigma) \right) b(r) \quad a = a_t \, dt + a_\sigma \, d\sigma
\]

with vanishing variation \( \delta b = 0 \) and allowed variations \( \delta a \neq 0 \)
Holonomies and boundary action

- locally Chern–Simons is trivial, but globally holonomies can exist
Holonomies and boundary action

- locally Chern–Simons is trivial, but globally holonomies can exist
- encode holonomies in (non-)periodicity properties of group element $g$

$$g(t, \sigma + 2\pi) = h g(t, \sigma) \quad h \in \text{SL}(2, \mathbb{R}) \quad \text{Tr} \ h = \text{Tr} \left( \mathcal{P} \exp \oint a_\sigma \, d\sigma \right)$$

assume for simplicity time-independence of $h$
Holonomies and boundary action

- locally Chern–Simons is trivial, but globally holonomies can exist
- encode holonomies in (non-)periodicity properties of group element \( g \)

\[
g(t, \sigma + 2\pi) = h g(t, \sigma) \quad h \in \text{SL}(2, \mathbb{R}) \quad \text{Tr } h = \text{Tr} \left( \mathcal{P} \exp \oint a_\sigma \, d\sigma \right)
\]

assume for simplicity time-independence of \( h \)

- Hamiltonian action decomposes into three terms

\[
I_{\text{CS}}[A] = -\frac{k}{4\pi} \int_{\partial \mathcal{M}} dt \, d\sigma \, \text{Tr} \left( g' g^{-1} \dot{g} g^{-1} \right) - \frac{k}{12\pi} \int_{\mathcal{M}} \text{Tr} \left( G^{-1} \, dG \right)^3 + I_{\partial \mathcal{M}}
\]
Holonomies and boundary action

- locally Chern–Simons is trivial, but globally holonomies can exist
- encode holonomies in (non-)periodicity properties of group element \( g \)

\[
g(t, \sigma + 2\pi) = h g(t, \sigma) \quad h \in \text{SL}(2, \mathbb{R}) \quad \text{Tr} \ h = \text{Tr} \left( \mathcal{P} \exp \int a_\sigma \, d\sigma \right)
\]

assume for simplicity time-independence of \( h \)

- Hamiltonian action decomposes into three terms

\[
I_{\text{CS}}[A] = -\frac{k}{4\pi} \int_{\partial \mathcal{M}} \, dt \, d\sigma \text{Tr} \left( g' \, g^{-1} \, \dot{g} \, g^{-1} \right) - \frac{k}{12\pi} \int_{\mathcal{M}} \text{Tr} \, (G^{-1} \, dG)^3 + I_{\partial \mathcal{M}}
\]

- Gauss decomposition \( G = e^{XL} + e^\Phi L_0^e Y L^- \) yields boundary action

\[
I_{\text{CS}}[\Phi, X, Y] = -\frac{k}{4\pi} \int_{\partial \mathcal{M}} \, dt \, d\sigma \left( \frac{1}{2} \dot{\Phi} \Phi' - 2e^\Phi X' \dot{Y} \right) + I_{\partial \mathcal{M}}
\]

used standard basis for \( \text{SL}(2, \mathbb{R}) \): \([L_n, L_m] = (n - m) \, L_{n+m} \) for \( n, m = 0, \pm 1 \)

also used Polyakov–Wiegmann identity to show \( b \)-independence of action and chose \( b = 1 \) at \( \partial \mathcal{M} \)
Outline

Overture

Hamiltonian reduction

Near horizon boundary conditions

Near horizon Hamiltonian

KdV deformation

Conclusions
Near horizon boundary conditions (metric formulation)

so far have not imposed any boundary conditions
so far have not imposed any boundary conditions

▶ consider near horizon expansion

\[ ds^2 = -\kappa^2 r^2 \, dt^2 + dr^2 + \frac{\ell^2}{4} \left( J^+ + J^- \right)^2 \, d\sigma^2 + \kappa \left( J^+ - J^- \right) r^2 \, dt \, d\sigma + \ldots \]

\( r \to 0 \): Rindler horizon
\( \kappa \): surface gravity
\( J^+(t, \sigma) + J^-(t, \sigma) \): metric transversal to horizon
\ldots: terms of higher order in \( r \)
Near horizon boundary conditions (metric formulation)

so far have not imposed any boundary conditions

- consider near horizon expansion

\[ ds^2 = -\kappa^2 r^2 \, dt^2 + dr^2 + \frac{\ell^2}{4} \left( J^+ + J^- \right)^2 \, d\sigma^2 + \kappa \left( J^+ - J^- \right) r^2 \, dt \, d\sigma + \ldots \]

\( r \to 0 \): Rindler horizon
\( \kappa \): surface gravity
\( J^+(t, \sigma) + J^-(t, \sigma) \): metric transversal to horizon
\( \ldots \): terms of higher order in \( r \)

- assumption 1: impose boundary conditions on (stretched) horizon, not at infinity

\[ J^+(t, \sigma) + J^-(t, \sigma) = \text{constant} \]

⇒ "holographic Ward identities" imply time-independence of state-dependent fct's

\[ \dot{J}^\pm = 0 \]
Near horizon boundary conditions (metric formulation)

so far have not imposed any boundary conditions

- consider near horizon expansion

\[ ds^2 = -\kappa^2 r^2 \, dt^2 + dr^2 + \frac{\ell^2}{4} \left( J^+ + J^- \right)^2 \, d\sigma^2 + \kappa \left( J^+ - J^- \right) r^2 \, dt \, d\sigma + \ldots \]

\( r \to 0 \): Rindler horizon
\( \kappa \): surface gravity
\( J^+(t, \sigma) + J^-(t, \sigma) \): metric transversal to horizon
\( \ldots \): terms of higher order in \( r \)

- assumption 1: impose boundary conditions on (stretched) horizon, not at infinity

- assumption 2: surface gravity state-independent, \( \delta \kappa = 0 \)
Near horizon boundary conditions (metric formulation)

so far have not imposed any boundary conditions
▶ consider near horizon expansion

\[ ds^2 = -\kappa^2 r^2 \, dt^2 + dr^2 + \frac{\ell^2}{4} \left( \mathcal{J}^+ + \mathcal{J}^- \right)^2 \, d\sigma^2 + \kappa \left( \mathcal{J}^+ - \mathcal{J}^- \right) r^2 \, dt \, d\sigma + \ldots \]

\( r \to 0 \): Rindler horizon
\( \kappa \): surface gravity
\( \mathcal{J}^+(t, \sigma) + \mathcal{J}^-(t, \sigma) \): metric transversal to horizon
\ldots: terms of higher order in \( r \)

▶ assumption 1: impose boundary conditions on (stretched) horizon, not at infinity
▶ assumption 2: surface gravity state-independent, \( \delta \kappa = 0 \)
▶ assumption 3: other metric functions state-dependent, \( \delta \mathcal{J}^\pm \neq 0 \)
Near horizon boundary conditions (metric formulation)

so far have not imposed any boundary conditions

- consider near horizon expansion

\[ ds^2 = -\kappa^2 r^2 \, dt^2 + dr^2 + \frac{\ell^2}{4} \left( J^+ + J^- \right)^2 \, d\sigma^2 + \kappa \left( J^+ - J^- \right) r^2 \, dt \, d\sigma + \ldots \]

\( r \to 0 \): Rindler horizon
\( \kappa \): surface gravity
\( J^+(t, \sigma) + J^-(t, \sigma) \): metric transversal to horizon
\( \ldots \): terms of higher order in \( r \)

- assumption 1: impose boundary conditions on (stretched) horizon, not at infinity

- assumption 2: surface gravity state-independent, \( \delta \kappa = 0 \)

- assumption 3: other metric functions state-dependent, \( \delta J^\pm \neq 0 \)

- simplifying assumption: constant surface gravity \( \Rightarrow \) “holographic Ward identities” imply time-independence of state-dependent fct’s

\[ \dot{J}^\pm = 0 \]
Black holes can be deformed into black flowers Afshar et al. 16

Horizon can get excited by area preserving shear-deformations

$k = 1$

$k = 2$

$k = 3$

$k = 4$

$k = 5$

$k = 6$
Near horizon Chern–Simons connection

- same boundary conditions in Chern–Simons language:

\[ a = (\mathcal{J}(\sigma) \ d\sigma - \kappa \ dt) \ L_0 \quad A = b^{-1} (d+a)b \]
Near horizon Chern–Simons connection

- same boundary conditions in Chern–Simons language:

\[ a = \left( \mathcal{J}(\sigma) \, d\sigma - \kappa \, dt \right) L_0 \quad A = b^{-1} \left( d+a \right) b \]

- boundary condition preserving gauge transformations \( \delta_\varepsilon a = d\varepsilon + [a, \varepsilon] \):

\[ \delta_\varepsilon \mathcal{J} = \eta' \quad \varepsilon = \eta L_0 + \ldots \]
Near horizon Chern–Simons connection

- same boundary conditions in Chern–Simons language:
  \[ a = \left( \mathcal{J}(\sigma) \; d\sigma - \kappa \; dt \right) L_0 \quad A = b^{-1} \left( d + a \right) b \]

- boundary condition preserving gauge transformations \( \delta \varepsilon a = d\varepsilon + [a, \varepsilon] \):
  \[ \delta \varepsilon J = \eta' \quad \varepsilon = \eta L_0 + \ldots \]

- canonical boundary charges in general:
  \[ \delta Q[\varepsilon] = -\frac{k}{2\pi} \oint d\sigma \; \text{Tr} \left( \varepsilon \delta a_{\sigma} \right) \]
Near horizon Chern–Simons connection

- same boundary conditions in Chern–Simons language:
  \[
  a = \left( \mathcal{J}(\sigma) \, d\sigma - \kappa \, dt \right) L_0 \quad A = b^{-1} \left( d + a \right) b
  \]

- boundary condition preserving gauge trasfos \( \delta_\epsilon a = d\epsilon + [a, \epsilon] \):
  \[
  \delta_\epsilon \mathcal{J} = \eta' \quad \epsilon = \eta L_0 + \ldots
  \]

- canonical boundary charges in general:
  \[
  \delta Q[\epsilon] = - \frac{k}{2\pi} \oint d\sigma \text{Tr} \left( \epsilon \delta a_\sigma \right)
  \]

- canonical boundary charges for near horizon boundary conditions:
  \[
  Q[\eta] = - \frac{k}{4\pi} \oint d\sigma \eta \mathcal{J}
  \]
Near horizon Chern–Simons connection

- same boundary conditions in Chern–Simons language:
  \[ a = \left( \mathcal{J}(\sigma) \, d\sigma - \kappa \, dt \right) L_0 \quad A = b^{-1} \left( d + a \right) b \]

- boundary condition preserving gauge transformations \( \delta_\varepsilon a = d\varepsilon + [a, \varepsilon] \):
  \[ \delta_\varepsilon \mathcal{J} = \eta' \quad \varepsilon = \eta L_0 + \ldots \]

- canonical boundary charges in general:
  \[ \delta Q[\varepsilon] = -\frac{k}{2\pi} \oint d\sigma \, \text{Tr} \left( \varepsilon \delta a_\sigma \right) \]

- canonical boundary charges for near horizon boundary conditions:
  \[ Q[\eta] = -\frac{k}{4\pi} \oint d\sigma \, \eta \mathcal{J} \]

- like Brown–Henneaux: 2 towers of conserved boundary charges \( \mathcal{J}^\pm \)
Near horizon symmetries

- near horizon symmetries = all boundary condition preserving trasfos modulo trivial gauge trasfos
Near horizon symmetries

- near horizon symmetries = all boundary condition preserving trasfos modulo trivial gauge trasfos
- near horizon symmetries generated by canonical boundary charges

\[ \delta_{\eta_1} Q[\eta_2] = \{Q[\eta_1], Q[\eta_2]\} = -\frac{k}{4\pi} \oint d\sigma \eta_2 \eta_1' \]
Near horizon symmetries

- near horizon symmetries $= \text{all boundary condition preserving trasfos modulo trivial gauge trasfos}
- near horizon symmetries generated by canonical boundary charges

$$\delta_{\eta_1} Q[\eta_2] = \{Q[\eta_1], Q[\eta_2]\} = -\frac{k}{4\pi} \oint d\sigma \eta_2 \eta_1'$$

- introduce Fourier modes

$$J_n = \frac{1}{2\pi} \oint d\sigma J e^{in\sigma}$$
Near horizon symmetries

- near horizon symmetries = all boundary condition preserving trasfos modulo trivial gauge trasfos
- near horizon symmetries generated by canonical boundary charges
  \[ \delta_{\eta_1} Q[\eta_2] = \{ Q[\eta_1], Q[\eta_2] \} = -\frac{k}{4\pi} \oint d\sigma \eta_2 \eta'_1 \]
- introduce Fourier modes
  \[ J_n = \frac{1}{2\pi} \oint d\sigma J e^{in\sigma} \]
- find two affine \( u(1) \) current algebras as near horizon symmetries
  \[ [J_n, J_m] = \frac{2}{k} n \delta_{n+m,0} \]

replaced Poisson brackets by commutators as usual, \( i\{ , \} \rightarrow [ , ] \); note: algebra isomorphic to Heisenberg algebras
Near horizon symmetries

- near horizon symmetries = all boundary condition preserving trasfos modulo trivial gauge trasfos
- near horizon symmetries generated by canonical boundary charges

\[ \delta_{\eta_1} Q[\eta_2] = \{ Q[\eta_1], Q[\eta_2] \} = -\frac{k}{4\pi} \oint d\sigma \eta_2 \eta_1' \]

- introduce Fourier modes

\[ J_n = \frac{1}{2\pi} \oint d\sigma J e^{in\sigma} \]

- find two affine $u(1)$ current algebras as near horizon symmetries

\[ [J_n, J_m] = \frac{2}{k} n \delta_{n+m,0} \]

replaced Poisson brackets by commutators as usual, $i\{ , \} \rightarrow [, ]$; note: algebra isomorphic to Heisenberg algebras

- simpler than Brown–Henneaux, who found Virasoros

the Brown–Henneaux Virasoros are recovered unambiguously through a twisted Sugawara-construction
Near horizon symmetries

- near horizon symmetries = all boundary condition preserving transformations modulo trivial gauge transformations
- near horizon symmetries generated by canonical boundary charges

\[ \delta_{\eta_1} Q[\eta_2] = \{ Q[\eta_1], Q[\eta_2] \} = -\frac{k}{4\pi} \oint d\sigma \eta_2 \eta_1' \]

- introduce Fourier modes

\[ J_n = \frac{1}{2\pi} \oint d\sigma J e^{in\sigma} \]

- find two affine $u(1)$ current algebras as near horizon symmetries

\[ [J_n, J_m] = \frac{2}{k} n \delta_{n+m,0} \]

replaced Poisson brackets by commutators as usual, $i\{,\} \rightarrow [,]$; note: algebra isomorphic to Heisenberg algebras

- simpler than Brown–Henneaux, who found Virasoros

the Brown–Henneaux Virasoros are recovered unambiguously through a twisted Sugawara-construction

- near-horizon (Cardy-like) entropy formula: \[ S = 2\pi \left( J_0^+ + J_0^- \right) \]
Unique features of near horizon boundary conditions

1. All states allowed by bc’s have same temperature

By contrast: asymptotically AdS or flat space bc’s allow for black hole states at different masses and hence different temperatures
Unique features of near horizon boundary conditions

1. All states allowed by bc’s have same temperature
2. All states allowed by bc’s are regular
   (in particular, they have no conical singularities at the horizon in the Euclidean formulation)

By contrast: for given temperature not all states in theories with asymptotically AdS or flat space bc’s are free from conical singularities; usually a unique black hole state is picked

5. Leads to soft Heisenberg hair (see next slide!)
Unique features of near horizon boundary conditions

1. All states allowed by bc’s have same temperature
2. All states allowed by bc’s are regular
   (in particular, they have no conical singularities at the horizon in the Euclidean formulation)
3. There is a non-trivial reducibility parameter (= Killing vector)

By contrast: for any other known (non-trivial) bc’s there is no vector field that is Killing for all geometries allowed by bc’s
Unique features of near horizon boundary conditions

1. All states allowed by bc’s have same temperature
2. All states allowed by bc’s are regular
   (in particular, they have no conical singularities at the horizon in the Euclidean formulation)
3. There is a non-trivial reducibility parameter (= Killing vector)

\[
A^\pm = b^{\mp 1} (d + a^\pm) b^{\pm 1}
\]
\[
a^\pm = L_0 \left( J^\pm \, d\sigma - \kappa \, dt \right)
\]
\[
b = \exp \left[ (L_+ - L_-) \frac{r}{2} \right]
\]

near horizon metric recovered from

\[
g_{\mu\nu} = \frac{\ell^2}{2} \, \text{Tr} \left( (A^+_{\mu} - A^-_{\mu})(A^+_{\nu} - A^-_{\nu}) \right)
\]
Unique features of near horizon boundary conditions

1. All states allowed by bc’s have same temperature
2. All states allowed by bc’s are regular
   (in particular, they have no conical singularities at the horizon in the Euclidean formulation)
3. There is a non-trivial reducibility parameter (\(=\) Killing vector)

\[
\begin{align*}
A^\pm &= b^{\pm1}(d+a^\pm)b^{\pm1} \\
a^\pm &= L_0(J^\pm d\sigma - \kappa dt) \\
b &= \exp[(L_+ - L_-)r/2]
\end{align*}
\]

near horizon metric recovered from

\[
g_{\mu\nu} = \frac{\ell^2}{2} \text{Tr} \left( (A^+_\mu - A^-_\mu)(A^+_\nu - A^-_\nu) \right)
\]

5. Leads to soft Heisenberg hair (see next slide!)
Soft Heisenberg hair

- Black flower excitations = hair of black holes
  Algebraically, excitations from descendants

\[
|\text{black flower}\rangle \sim \prod_{n_i^+ > 0} J^+_{n_i^+} J^-_{n_i^-} |\text{black hole}\rangle
\]
Soft Heisenberg hair

- Black flower excitations = hair of black holes
  Algebraically, excitations from descendants
  \[
  |\text{black flower}\rangle \sim \prod_{n_i > 0} J^+_\pm n_i J^-_{-n_i} |\text{black hole}\rangle
  \]

- What is energy of such excitations?
Soft Heisenberg hair

- Black flower excitations = hair of black holes
  Algebraically, excitations from descendants

  $$|\text{black flower}\rangle \sim \prod_{n_i > 0} J^+_{-n_i} J^-_{-n_i} |\text{black hole}\rangle$$

- What is energy of such excitations?

- Near horizon Hamiltonian = boundary charge associated with unit time-translations*

  $$H = Q[\partial_t] = \kappa \left( J^+_0 + J^-_0 \right)$$

  commutes with all generators $J^\pm_n$

* units defined by specifying $\kappa$
Soft Heisenberg hair

- Black flower excitations = hair of black holes
  Algebraically, excitations from descendants
  \[
  \langle \text{black flower} \rangle \sim \prod_{n_i > 0} J^+_n J^-_n \langle \text{black hole} \rangle
  \]

- What is energy of such excitations?
- Near horizon Hamiltonian = boundary charge associated with unit time-translations
  \[
  H = Q[\partial_t] = \kappa (J^+_0 + J^-_0)
  \]
  commutes with all generators \( J^\pm_n \)

- \( H \)-eigenvalue of black flower = \( H \)-eigenvalue of black hole
Soft Heisenberg hair

- Black flower excitations = hair of black holes
  Algebraically, excitations from descendants
  
  \[ |\text{black flower}\rangle \sim \prod_{n_i^+ > 0} J_{-n_i}^+ J_{-n_i}^- |\text{black hole}\rangle \]

- What is energy of such excitations?
- Near horizon Hamiltonian = boundary charge associated with unit time-translations
  \[ H = Q[\partial_t] = \kappa (J_0^+ + J_0^-) \]
  commutes with all generators \( J_n^{\pm} \)

- \( H \)-eigenvalue of black flower = \( H \)-eigenvalue of black hole
- Black flower excitations do not change energy of black hole!
Soft Heisenberg hair

- Black flower excitations = hair of black holes
  Algebraically, excitations from descendants
  \[ |\text{black flower}\rangle \sim \prod_{n_i>0} J^+_{-n_i} J^-_{-n_i} |\text{black hole}\rangle \]

- What is energy of such excitations?
- Near horizon Hamiltonian = boundary charge associated with unit time-translations
  \[ H = Q[\partial_t] = \kappa (J^+_0 + J^-_0) \]
  commutes with all generators \( J^\pm_n \)
- \( H \)-eigenvalue of black flower = \( H \)-eigenvalue of black hole
- Black flower excitations do not change energy of black hole!

Black flower excitations = soft hair in sense of Hawking, Perry and Strominger ’16
Soft Heisenberg hair

- Black flower excitations = hair of black holes
  Algebraically, excitations from descendants
  \[ |\text{black flower}\rangle \sim \prod_{n_i>0} J^+_{-n_i} J^-_{-n_i} |\text{black hole}\rangle \]

- What is energy of such excitations?
- Near horizon Hamiltonian = boundary charge associated with unit time-translations
  \[ H = Q[\partial_t] = \kappa (J^+_0 + J^-_0) \]
  commutes with all generators \( J^\pm_n \)
  \( H \)-eigenvalue of black flower = \( H \)-eigenvalue of black hole
  Black flower excitations do not change energy of black hole!

Black flower excitations = soft hair in sense of Hawking, Perry and Strominger ’16
Call it “soft Heisenberg hair”
Outline

Overture

Hamiltonian reduction

Near horizon boundary conditions

Near horizon Hamiltonian

KdV deformation

Conclusions
Near horizon boundary action

▶ recall general boundary action

\[
I_{CS}[\Phi, X, Y] = -\frac{k}{4\pi} \int_{\partial M} dt \, d\sigma \left( \frac{1}{2} \dot{\Phi} \dot{\Phi}' - 2e^\Phi X' \dot{Y} \right) + I_{\partial M}
\]
Near horizon boundary action

- recall general boundary action

\[ I_{CS}[\Phi, X, Y] = -\frac{k}{4\pi} \int_{\partial M} dt \, d\sigma \left( \frac{1}{2} \dot{\Phi} \Phi' - 2e^\Phi X' \dot{Y} \right) + I_{\partial M} \]

- near horizon boundary conditions imply

\[ \Phi' = \mathcal{J} \quad X' = 0 \]
Near horizon boundary action

- recall general boundary action

\[ I_{CS}[\Phi, X, Y] = -\frac{k}{4\pi} \int_{\partial M} dt \, d\sigma \left( \frac{1}{2} \dot{\Phi} \Phi' - 2e^{\Phi} X' \dot{Y} \right) + I_{\partial M} \]

- near horizon boundary conditions imply

\[ \Phi' = \mathcal{J} \quad X' = 0 \]

- scalar field \( \Phi \) has generalized periodicity property

\[ \Phi(t, \sigma + 2\pi) = \Phi(t, \sigma) + 2\pi J_0 \]
Near horizon boundary action

- recall general boundary action

\[
I_{\text{CS}}[\Phi, X, Y] = -\frac{k}{4\pi} \int_{\partial M} dt \, d\sigma \left( \frac{1}{2} \dot{\Phi} \Phi' - 2e^\Phi X' \dot{Y} \right) + I_{\partial M}
\]

- near horizon boundary conditions imply

\[
\Phi' = J \quad X' = 0
\]

- scalar field \( \Phi \) has generalized periodicity property

\[
\Phi(t, \sigma + 2\pi) = \Phi(t, \sigma) + 2\pi J_0
\]

- near horizon boundary action simplifies

\[
I_{\text{CS}}[\Phi] = -\frac{k}{4\pi} \int_{\partial M} dt \, d\sigma \frac{1}{2} \dot{\Phi} \Phi' + I_{\partial M}
\]
Near horizon boundary action

- recall general boundary action

\[ I_{CS}[\Phi, X, Y] = -\frac{k}{4\pi} \int_{\partial\mathcal{M}} dt \, d\sigma \left( \frac{1}{2} \dot{\Phi} \Phi' - 2e^\Phi X' \dot{Y} \right) + I_{\partial\mathcal{M}} \]

- near horizon boundary conditions imply

\[ \Phi' = \mathcal{J} \quad \quad X' = 0 \]

- scalar field \( \Phi \) has generalized periodicity property

\[ \Phi(t, \sigma + 2\pi) = \Phi(t, \sigma) + 2\pi J_0 \]

- near horizon boundary action simplifies

\[ I_{CS}[\Phi] = -\frac{k}{4\pi} \int_{\partial\mathcal{M}} dt \, d\sigma \frac{1}{2} \dot{\Phi} \Phi' + I_{\partial\mathcal{M}} \]

- still need to discuss \( I_{\partial\mathcal{M}} \), since it encodes the boundary Hamiltonian!
Simplest choice of boundary term

- well-defined variational principle if

\[ \delta I_{\partial \mathcal{M}} = \frac{k}{2\pi} \int_{\partial \mathcal{M}} dt \, d\sigma \, Tr \left( a_t \delta a_\sigma \right) \]
Simplest choice of boundary term

- well-defined variational principle if
  \[ \delta I_{\partial M} = \frac{k}{2\pi} \int_{\partial M} dt \, d\sigma \, \text{Tr} \left( a_t \delta a_\sigma \right) \]

- defining \( a_t = -\zeta(t, \sigma) L_0 \) and using near horizon boundary conditions for \( a_\sigma \) yields
  \[ \delta I_{\partial M} = \frac{k}{2\pi} \int_{\partial M} dt \, d\sigma \, \zeta \, \delta J \]
Simplest choice of boundary term

- well-defined variational principle if

$$\delta I_{\partial \mathcal{M}} = \frac{k}{2\pi} \int_{\partial \mathcal{M}} dt \, d\sigma \, \text{Tr} \left( a_t \delta a_\sigma \right)$$

- defining $a_t = -\zeta(t, \sigma) L_0$ and using near horizon boundary conditions for $a_\sigma$ yields

$$\delta I_{\partial \mathcal{M}} = \frac{k}{2\pi} \int_{\partial \mathcal{M}} dt \, d\sigma \, \zeta \, \delta J$$

- integrability of boundary action requires

$$\zeta(J) = \frac{\delta \mathcal{H}}{\delta J}$$

where $\mathcal{H}$ is the boundary Hamiltonian density
Simplest choice of boundary term

- well-defined variational principle if
  \[ \delta I_{\partial M} = \frac{k}{2\pi} \int_{\partial M} dt \, d\sigma \, \text{Tr} \left( a_t \delta a_\sigma \right) \]

- defining \( a_t = -\zeta(t, \sigma) L_0 \) and using near horizon boundary conditions for \( a_\sigma \) yields
  \[ \delta I_{\partial M} = \frac{k}{2\pi} \int_{\partial M} dt \, d\sigma \, \zeta \, \delta J \]

- integrability of boundary action requires
  \[ \zeta(J) = \frac{\delta \mathcal{H}}{\delta J} \]
  where \( \mathcal{H} \) is the boundary Hamiltonian density

- simplest choice (near horizon boundary conditions for \( a_t \)):
  \[ \delta \zeta = 0 \]

  make this choice to obtain near horizon Hamiltonian!
Near horizon Hamiltonian

- solving integrability condition

\[ \zeta (J) = \frac{\delta \mathcal{H}}{\delta J} \]

for \( \mathcal{H} \) yields boundary Hamiltonian density

\[ \mathcal{H}_{\text{NH}} = \zeta J = \zeta \Phi' \]
Near horizon Hamiltonian

- solving integrability condition

\[ \zeta(J) = \frac{\delta H}{\delta J} \]

for \( H \) yields boundary Hamiltonian density

\[ \mathcal{H}_{NH} = \zeta J = \zeta \Phi' \]

- this was the main result announced in the beginning
Near horizon Hamiltonian

- solving integrability condition

\[ \zeta(J) = \frac{\delta \mathcal{H}}{\delta J} \]

for \( \mathcal{H} \) yields boundary Hamiltonian density

\[ \mathcal{H}_{NH} = \zeta J = \zeta \Phi' \]

- this was the main result announced in the beginning

- full boundary action given by

\[ I_{NH}[\Phi] = -\frac{k}{2\pi} \int dt \, d\sigma \left( \frac{1}{2} \dot{\Phi} \Phi' + \zeta \Phi' \right) \]

\[ \Rightarrow \text{momentum given by spatial derivative, } \Pi \sim \Phi'! \]
Near horizon Hamiltonian

- solving integrability condition

\[ \zeta(J) = \frac{\delta H}{\delta J} \]

for \( H \) yields boundary Hamiltonian density

\[ H_{NH} = \zeta J = \zeta \Phi' \]

- this was the main result announced in the beginning

- full boundary action given by

\[ I_{NH}[\Phi] = -\frac{k}{2\pi} \int dt \, d\sigma \left( \frac{1}{2} \dot{\Phi} \Phi' + \zeta \Phi' \right) \]

\( \Rightarrow \) momentum given by spatial derivative, \( \Pi \sim \Phi' \)

- near horizon Hamiltonian given by zero mode generator

\[ H_{NH} = \frac{k}{2\pi} \oint d\sigma \, H_{NH} = \frac{k}{2} \zeta J_0 \]

recovers result expected from near horizon symmetry analysis
Mode decomposition

- near horizon equations of motion

\[ \dot{\Phi}' = 0 \]

solved by

\[ \Phi(t, \sigma) \big|_{\text{EOM}} = \Phi_0(t) + J_0 \sigma + \sum_{n \neq 0} \frac{J_n}{in} e^{in\sigma} \]
Mode decomposition

- near horizon equations of motion

\[ \dot{\Phi}' = 0 \]

solved by

\[ \Phi(t, \sigma) \bigg|_{\text{EOM}} = \Phi_0(t) + J_0 \sigma + \sum_{n \neq 0} \frac{J_n}{in} e^{in\sigma} \]

- off-shell similar mode-decomposition

\[ \Phi(t, \sigma) = \Phi_0(t) + J_0(t) \sigma + \sum_{n \neq 0} \frac{J_n(t)}{in} e^{in\sigma} \]

due to generalized periodicity property of \( \Phi \)
Mode decomposition

- near horizon equations of motion

\[ \dot{\Phi}' = 0 \]

solved by

\[ \Phi(t, \sigma) \bigg|_{EOM} = \Phi_0(t) + J_0 \sigma + \sum_{n \neq 0} \frac{J_n}{i n} e^{i n \sigma} \]

- off-shell similar mode-decomposition

\[ \Phi(t, \sigma) = \Phi_0(t) + J_0(t) \sigma + \sum_{n \neq 0} \frac{J_n(t)}{i n} e^{i n \sigma} \]

due to generalized periodicity property of \( \Phi \)

- time-independence of holonomy requires \( \dot{J}_0 = 0 \)
Mode decomposition

- near horizon equations of motion
  \[ \dot{\Phi}' = 0 \]
  solved by
  \[ \Phi(t, \sigma)|_{\text{EOM}} = \Phi_0(t) + J_0 \sigma + \sum_{n \neq 0} \frac{J_n}{in} e^{in\sigma} \]

- off-shell similar mode-decomposition
  \[ \Phi(t, \sigma) = \Phi_0(t) + J_0(t) \sigma + \sum_{n \neq 0} \frac{J_n(t)}{in} e^{in\sigma} \]
  due to generalized periodicity property of \( \Phi \)

- time-independence of holonomy requires \( \dot{J}_0 = 0 \)

- off-shell mode-decomposition in near horizon boundary action:
  \[ I_{\text{NH}}[\Phi_0, J_n] = \frac{k}{2} \int dt \left( -\frac{1}{2} \dot{\Phi}_0 J_0 + \sum_{n > 0} i \frac{n}{i} \dot{J}_n J_{-n} - \zeta J_0 \right) \]
Florenini–Jackiw symplectic structure

reminder:

\[ I_{\text{NH}}[\Phi_0, J_n] = \frac{k}{2} \int dt \left( -\frac{1}{2} \dot{\Phi}_0 J_0 + \sum_{n>0} \frac{i}{n} \dot{J}_n J_{-n} - \zeta J_0 \right) \]

▷ rewrite near horizon boundary action in canonical form

\[ I_{\text{NH}}[\Phi_0, J_n] = \int dt \left( \dot{\Phi}_0 \Pi_0 + \sum_{n>0} \dot{J}_n \Pi_n - H_{\text{NH}} \right) \]
Florenini–Jackiw symplectic structure

reminder:

\[ I_{\text{NH}}[\Phi_0, J_n] = \frac{k}{2} \int dt \left( -\frac{1}{2} \dot{\Phi}_0 J_0 + \sum_{n>0} \frac{i}{n} \dot{J}_n J_{-n} - \zeta J_0 \right) \]

▶ rewrite near horizon boundary action in canonical form

\[ I_{\text{NH}}[\Phi_0, J_n] = \int dt \left( \dot{\Phi}_0 \Pi_0 + \sum_{n>0} \dot{J}_n \Pi_n - H_{\text{NH}} \right) \]

▶ yields relations for momenta (see e.g. Faddeev–Jackiw)

\[ \Pi_0 = -\frac{k}{4} J_0 \quad \Pi_n = \frac{ik}{2n} J_{-n} \]
Floreanini–Jackiw symplectic structure

- rewrite near horizon boundary action in canonical form

\[ I_{NH}[\Phi_0, J_n] = \int dt \left( \dot{\Phi}_0 \Pi_0 + \sum_{n>0} \dot{J}_n \Pi_n - H_{NH} \right) \]

- yields relations for momenta (see e.g. Faddeev–Jackiw)

\[ \Pi_0 = -\frac{k}{4} J_0 \quad \Pi_n = \frac{ik}{2n} J_{-n} \]

- canonical Poisson brackets \( \{ \Phi_0, \Pi_0 \} = 1, \{ J_n, \Pi_m \} = \delta_{n,m} \) recover precisely near horizon symmetry algebra

\[ i\{ J_n, J_m \} = \frac{2}{k} n \delta_{n+m,0} \]

plus an extra relation

\[ i\{ J_0, \Phi_0 \} = \frac{4i}{k} \]
Floreanini–Jackiw symplectic structure

- rewrite near horizon boundary action in canonical form

\[ I_{NH}[\Phi_0, J_n] = \int dt \left( \dot{\Phi}_0 \Pi_0 + \sum_{n>0} \dot{J}_n \Pi_n - H_{NH} \right) \]

- yields relations for momenta (see e.g. Faddeev–Jackiw)

\[ \Pi_0 = -\frac{k}{4} J_0 \quad \Pi_n = \frac{ik}{2n} J_{-n} \]

- canonical Poisson brackets \( \{ \Phi_0, \Pi_0 \} = 1, \{ J_n, \Pi_m \} = \delta_{n,m} \) recover precisely near horizon symmetry algebra

\[ i\{J_n, J_m\} = \frac{2}{k} n \delta_{n+m,0} \]

plus an extra relation

\[ i\{J_0, \Phi_0\} = \frac{4i}{k} J_0 \]

- Hamiltonian \( H_{NH} \sim J_0 \) commutes with all canonical variables \( \Rightarrow \) expected softness property recovered!
Outline

Overture

Hamiltonian reduction

Near horizon boundary conditions

Near horizon Hamiltonian

KdV deformation

Conclusions
Other choices for boundary action

- would like to lift soft hair degeneracy
  - reason 1: because it allows to recover Brown–Henneaux story
  - reason 2: because lifting soft hair degeneracy may help to address the fantasy that soft hair excitations could correspond to black hole microstates (at least in semi-classical limit and far away from extremality)
  - reason 3: because we can and it is fun

idea: generalize near horizon boundary conditions and then take suitable limit approaching them again

achieve this by making “chemical potentials” state-dependent

ζ = ζ(J)

not unqiue how to deform; infinitely many possibilities

make particular choice to maintain certain scaling symmetries

start by recovering Brown–Henneaux boundary conditions and the Schwarzian action
Other choices for boundary action

- would like to lift soft hair degeneracy
  - reason 1: because it allows to recover Brown–Henneaux story
  - reason 2: because lifting soft hair degeneracy may help to address the fantasy that soft hair excitations could correspond to black hole microstates (at least in semi-classical limit and far away from extremality)
  - reason 3: because we can and it is fun

- idea: generalize near horizon boundary conditions and then take suitable limit approaching them again
Other choices for boundary action

- would like to lift soft hair degeneracy
  - reason 1: because it allows to recover Brown–Henneaux story
  - reason 2: because lifting soft hair degeneracy may help to address the fantasy that soft hair excitations could correspond to black hole microstates (at least in semi-classical limit and far away from extremality)
  - reason 3: because we can and it is fun

- idea: generalize near horizon boundary conditions and then take suitable limit approaching them again

- achieve this by making “chemical potentials” state-dependent

$$\zeta = \zeta(J)$$
Other choices for boundary action

- would like to lift soft hair degeneracy
  - reason 1: because it allows to recover Brown–Henneaux story
  - reason 2: because lifting soft hair degeneracy may help to address the fantasy that soft hair excitations could correspond to black hole microstates (at least in semi-classical limit and far away from extremality)
  - reason 3: because we can and it is fun

- idea: generalize near horizon boundary conditions and then take suitable limit approaching them again

- achieve this by making “chemical potentials” state-dependent

\[ \zeta = \zeta(\mathcal{J}) \]

- not unique how to deform; infinitely many possibilities
Other choices for boundary action

- would like to lift soft hair degeneracy
  - reason 1: because it allows to recover Brown–Henneaux story
  - reason 2: because lifting soft hair degeneracy may help to address the fantasy that soft hair excitations could correspond to black hole microstates (at least in semi-classical limit and far away from extremality)
  - reason 3: because we can and it is fun
- idea: generalize near horizon boundary conditions and then take suitable limit approaching them again
- achieve this by making “chemical potentials” state-dependent

\[ \zeta = \zeta(J) \]

- not unique how to deform; infinitely many possibilities
- make particular choice to maintain certain scaling symmetries
Other choices for boundary action

- would like to lift soft hair degeneracy
  - reason 1: because it allows to recover Brown–Henneaux story
  - reason 2: because lifting soft hair degeneracy may help to address the fantasy that soft hair excitations could correspond to black hole microstates (at least in semi-classical limit and far away from extremality)
  - reason 3: because we can and it is fun

- idea: generalize near horizon boundary conditions and then take suitable limit approaching them again

- achieve this by making “chemical potentials” state-dependent

\[ \zeta = \zeta(\mathcal{I}) \]

- not unique how to deform; infinitely many possibilities
- make particular choice to maintain certain scaling symmetries
- start by recovering Brown–Henneaux boundary conditions and the Schwarzian action
Recovering Brown–Henneaux and the Schwarzian action

- choose (with $\delta \mu = 0$)

\[ \zeta = \mu' - \mathcal{J} \mu \]
Recovering Brown–Henneaux and the Schwarzian action

- choose (with $\delta \mu = 0$)
  \[ \zeta = \mu' - \mathcal{J} \mu \]

- boundary term still integrable

\[
I_{\text{BH}}[\Phi] = \frac{k}{4\pi} \int dt \, d\sigma \, \mu \left( \frac{1}{2} \mathcal{J}^2 + \mathcal{J}' \right) = \frac{k}{2\pi} \int dt \, d\sigma \, \mu \mathcal{L}
\]

with

\[
\mathcal{L} = \frac{1}{4} \mathcal{J}^2 + \frac{1}{2} \mathcal{J}'
\]
Recovering Brown–Henneaux and the Schwarzian action

- Choose (with $\delta \mu = 0$)
  \[\zeta = \mu' - J\mu\]

- Boundary term still integrable
  \[I_{BH}[\Phi] = \frac{k}{4\pi} \int dt \, d\sigma \mu \left(\frac{1}{2} J^2 + J'\right) = \frac{k}{2\pi} \int dt \, d\sigma \mu \mathcal{L}\]
  with
  \[\mathcal{L} = \frac{1}{4} J^2 + \frac{1}{2} J'\]

- Boundary action analogous, but Hamiltonian density changes
  \[\mathcal{H}_{BH} = -\frac{k\mu}{8\pi} ((\Phi')^2 + 2\Phi'')\]

No longer have soft hair, since $\mathcal{H}_{BH}$ is not a boundary term and the associated Hamiltonian does not commute with all generators of the asymptotic symmetries!
Recovering Brown–Henneaux and the Schwarzian action

- choose (with $\delta \mu = 0$)
  \[ \zeta = \mu' - \mathcal{J} \mu \]

- boundary term still integrable
  \[
  I_{\text{BH}}[\Phi] = \frac{k}{4\pi} \int dt \, d\sigma \, \mu \left( \frac{1}{2} \mathcal{J}^2 + \mathcal{J}' \right) = \frac{k}{2\pi} \int dt \, d\sigma \, \mu \mathcal{L}
  \]

  with
  \[
  \mathcal{L} = \frac{1}{4} \mathcal{J}^2 + \frac{1}{2} \mathcal{J}'
  \]

- boundary action analogous, but Hamiltonian density changes
  \[
  \mathcal{H}_{\text{BH}} = -\frac{k\mu}{8\pi} \left( (\Phi')^2 + 2\Phi'' \right)
  \]

- expressing action instead in terms of $X' = e^{-\Phi}$ yields
  \[
  I_{\text{BH}}[X] = \frac{k}{4\pi} \int dt \, d\sigma \left( \frac{\dot{X}''}{X'} - \frac{3}{2} \frac{X'' \dot{X}'}{X'^2} - \mu \{X, \sigma\}_{\text{Sch}} \right)
  \]

  = geometric action of Virasoro group on coadjoint orbit
KdV integrable hierarchy

hierarchy of Hamiltonians:
  ▶ near horizon boundary conditions: \( H_0 \sim \oint d\sigma J \)
KdV integrable hierarchy

hierarchy of Hamiltonians:

- near horizon boundary conditions: $H_0 \sim \oint d\sigma \mathcal{J}$
- Brown–Henneaux: $H_1 \sim \oint d\sigma \mathcal{J}^2$

$R^{N+1}_{N}$ is a Gelfand–Dikii differential polynomial:

$$R^{N+1}_{N} = N + 1 \frac{d}{d N} R^N_D = J' + 2 JJ' + \frac{1}{2} J'' + \frac{1}{6} J''''$$

for $N = 2$ field equations are KdV equation:

$$\dot{J} = 2 JJ' + \frac{1}{3} J''$$

for general $N$ Hamiltonian density reads:

$$H_N \sim \frac{1}{N+1} J^{N+1} + \frac{N-1}{2} \sum_{i=1}^{N} h_i,N J^{N-i} \left( \partial_i \sigma J \right)^2 + H_{nl,N} J = \Phi'$$

non-linear term in derivatives $H_{nl,N}$ exists only for $N \geq 5$; the $h_i,N$ are computable rational coefficients.
KdV integrable hierarchy

hierarchy of Hamiltonians:

- near horizon boundary conditions: \( H_0 \sim \oint d\sigma J \)
- Brown–Henneaux: \( H_1 \sim \oint d\sigma J^2 \)
- KdV generalization:

\[
H_N \sim \oint d\sigma R_{N+1}(J)
\]

where \( R_{N+1} \) is a Gelfand–Dikii differential polynomial:

\[
R'_{N+1} = \frac{N+1}{2N+1} \mathcal{D} R_N \quad \mathcal{D} := J' + 2J \partial_\sigma + \frac{1}{2} \partial_\sigma^3
\]
KdV integrable hierarchy

hierarchy of Hamiltonians:

- near horizon boundary conditions: \( H_0 \sim \oint d\sigma \mathcal{J} \)
- Brown–Henneaux: \( H_1 \sim \oint d\sigma \mathcal{J}^2 \)
- KdV generalization:

\[
H_N \sim \oint d\sigma R_{N+1}(\mathcal{J})
\]

where \( R_{N+1} \) is a Gelfand–Dikii differential polynomial:

\[
R_{N+1}' = \frac{N+1}{2N+1} \mathcal{D} R_N \quad \mathcal{D} := \mathcal{J}' + 2\mathcal{J} \partial_\sigma + \frac{1}{2} \partial_\sigma^3
\]

- for \( N = 2 \) field equations are KdV equation

\[
\dot{\mathcal{J}} = 2\mathcal{J} \mathcal{J}' + \frac{1}{3} \mathcal{J}'''
\]
KdV integrable hierarchy

hierarchy of Hamiltonians:

- near horizon boundary conditions: \( H_0 \sim \oint d\sigma \mathcal{J} \)
- Brown–Henneaux: \( H_1 \sim \oint d\sigma \mathcal{J}^2 \)
- KdV generalization:

\[
H_N \sim \oint d\sigma R_{N+1}(\mathcal{J})
\]

where \( R_{N+1} \) is a Gelfand–Dikii differential polynomial:

\[
R'_{N+1} = \frac{N+1}{2N+1} \mathcal{D} R_N \quad \mathcal{D} := J' + 2J \partial_\sigma + \frac{1}{2} \partial_\sigma^3
\]

- for \( N = 2 \) field equations are KdV equation

\[
\dot{\mathcal{J}} = 2\mathcal{J} \mathcal{J}' + \frac{1}{3} \mathcal{J}'''
\]

- for general \( N \) Hamiltonian density reads

\[
\mathcal{H}_N \sim \frac{1}{N+1} \mathcal{J}^{N+1} + \sum_{i=1}^{N-1} h_{i,N} \mathcal{J}^{N-i-1}(\partial_\sigma^i \mathcal{J})^2 + \mathcal{H}_{N}^{nl} \quad \mathcal{J} = \Phi'
\]

non-linear term in derivatives \( \mathcal{H}_{N}^{nl} \) exists only for \( N \geq 5 \); the \( h_{i,N} \) are computable rational coefficients
Scaling properties

- for $N > 1$ field equations have anisotropic scale invariance

$$t \rightarrow \lambda^{2N-1} t \quad \sigma \rightarrow \lambda \sigma \quad \Phi \rightarrow \lambda^{-1} \Phi$$

action not invariant
Scaling properties

- for $N > 1$ field equations have anisotropic scale invariance
  \[ t \rightarrow \lambda^{2N-1} t \quad \sigma \rightarrow \lambda \sigma \quad \Phi \rightarrow \lambda^{-1} \Phi \]

  action not invariant

- for $N \leq 1$ field equations and action invariant under
  \[ t \rightarrow \lambda^N t \quad \sigma \rightarrow \lambda \sigma \quad \Phi \rightarrow \Phi \]
Scaling properties

- for $N > 1$ field equations have anisotropic scale invariance

$$t \to \lambda^{2N-1} t \quad \sigma \to \lambda \sigma \quad \Phi \to \lambda^{-1} \Phi$$

action not invariant

- for $N \leq 1$ field equations and action invariant under

$$t \to \lambda^N t \quad \sigma \to \lambda \sigma \quad \Phi \to \Phi$$

- we are interested in taking the limit $N \to 0^+$
Scaling properties

- for $N > 1$ field equations have anisotropic scale invariance

$$ t \rightarrow \lambda^{2N-1} t \quad \sigma \rightarrow \lambda \sigma \quad \Phi \rightarrow \lambda^{-1} \Phi $$

action not invariant

- for $N \leq 1$ field equations and action invariant under

$$ t \rightarrow \lambda^N t \quad \sigma \rightarrow \lambda \sigma \quad \Phi \rightarrow \Phi $$

- we are interested in taking the limit $N \rightarrow 0^+$

- analytically continue $N \in [0, 1]$, keeping scale invariance of action
Scaling properties

- for $N > 1$ field equations have anisotropic scale invariance
  \[ t \rightarrow \lambda^{2N-1} t \quad \sigma \rightarrow \lambda \sigma \quad \Phi \rightarrow \lambda^{-1} \Phi \]

  action not invariant

- for $N \leq 1$ field equations and action invariant under
  \[ t \rightarrow \lambda^N t \quad \sigma \rightarrow \lambda \sigma \quad \Phi \rightarrow \Phi \]

- we are interested in taking the limit $N \rightarrow 0^+$

- analytically continue $N \in [0, 1]$, keeping scale invariance of action

- consider continuous family of boundary Hamiltonians ($\varepsilon \in [0, 1]$)

  \[
  H_\varepsilon = \frac{k}{4\pi} \frac{\zeta_\varepsilon}{\varepsilon(1 + \varepsilon)} \oint d\sigma \mathcal{J}^{1+\varepsilon}
  \]
Scaling properties

▶ for $N > 1$ field equations have anisotropic scale invariance

$$t \to \lambda^{2N-1} t \quad \sigma \to \lambda \sigma \quad \Phi \to \lambda^{-1} \Phi$$

action not invariant

▶ for $N \leq 1$ field equations and action invariant under

$$t \to \lambda^N t \quad \sigma \to \lambda \sigma \quad \Phi \to \Phi$$

▶ we are interested in taking the limit $N \to 0^+$

▶ analytically continue $N \in [0, 1]$, keeping scale invariance of action

▶ consider continuous family of boundary Hamiltonians ($\varepsilon \in [0, 1]$)

$$H_\varepsilon = \frac{k}{4\pi} \frac{\zeta_\varepsilon}{\varepsilon(1 + \varepsilon)} \oint d\sigma \mathcal{J}^{1+\varepsilon}$$

▶ note that we rescaled by $1/\varepsilon$ to have non-trivial limit $\varepsilon \to 0^+$!
KdV scaling limit for near horizon Hamiltonian

- take now the limit $\varepsilon \to 0^+$

$$H_{\log} := \lim_{\varepsilon \to 0^+} H_\varepsilon = \frac{k \zeta_\varepsilon}{4\pi} \oint d\sigma \mathcal{J} \ln \mathcal{J}$$
KdV scaling limit for near horizon Hamiltonian

- take now the limit $\varepsilon \rightarrow 0^+$

$$H_{\log} := \lim_{\varepsilon \rightarrow 0^+} H_\varepsilon = \frac{k\zeta_\varepsilon}{4\pi} \oint d\sigma \mathcal{J} \ln \mathcal{J}$$

- limiting boundary action reads

$$I_{\log}[\Phi] = -\frac{k}{4\pi} \int dt d\sigma \left( \frac{1}{2} \dot{\Phi} \Phi' + \zeta_\varepsilon \Phi' \ln (\Phi') \right)$$
KdV scaling limit for near horizon Hamiltonian

- take now the limit $\varepsilon \to 0^+$

$$ H_{\log} := \lim_{\varepsilon \to 0^+} H_{\varepsilon} = \frac{k \zeta_{\varepsilon}}{4\pi} \oint d\sigma \mathcal{J} \ln \mathcal{J} $$

- limiting boundary action reads

$$ I_{\log}[\Phi] = -\frac{k}{4\pi} \int dt d\sigma \left( \frac{1}{2} \dot{\Phi} \Phi' + \zeta_{\varepsilon} \Phi' \ln (\Phi') \right) $$

- field equations

$$ \dot{\Phi}' = -\zeta_{\varepsilon} \frac{\Phi''}{\Phi'} $$

yield simple solution for modes in limit of large $J_0$
KdV scaling limit for near horizon Hamiltonian

- take now the limit $\epsilon \to 0^+$

$$H_{\log} := \lim_{\epsilon \to 0^+} H_\epsilon = \frac{k\zeta_\epsilon}{4\pi} \oint d\sigma J \ln J$$

- limiting boundary action reads

$$I_{\log}[\Phi] = -\frac{k}{4\pi} \int dt d\sigma \left( \frac{1}{2} \dot{\Phi} \Phi' + \zeta_\epsilon \Phi' \ln (\Phi') \right)$$

- field equations

$$\dot{\Phi}' = -\zeta_\epsilon \frac{\Phi''}{\Phi'}$$

yield simple solution for modes in limit of large $J_0$

- in that limit boundary action reads

$$I_{\log}[\Phi_0, J_n, \Pi_n] = \int dt \left( \dot{\Phi}_0 \Pi_0 + \sum_{n>0} \dot{J}_n \Pi_n - \frac{ik\zeta_\epsilon}{4\Pi_0} \sum_{n>0} n\Pi_n J_n \right)$$
KdV scaling limit for near horizon Hamiltonian

- take now the limit $\varepsilon \to 0^+$

$$H_{\log} := \lim_{\varepsilon \to 0^+} H_\varepsilon = \frac{k \zeta_\varepsilon}{4\pi} \oint d\sigma \mathcal{J} \ln \mathcal{J}$$

- limiting boundary action reads

$$I_{\log}[\Phi] = -\frac{k}{4\pi} \int dt d\sigma \left( \frac{1}{2} \ddot{\Phi} \Phi' + \zeta_\varepsilon \Phi' \ln (\Phi') \right)$$

- field equations

$$\dot{\Phi}' = -\zeta_\varepsilon \frac{\Phi''}{\Phi'}$$

yield simple solution for modes in limit of large $J_0$

- in that limit boundary action reads

$$I_{\log}[\Phi_0, J_n, \Pi_n] = \int dt \left( \dot{\Phi}_0 \Pi_0 + \sum_{n>0} \dot{J}_n \Pi_n - \frac{ik\zeta_\varepsilon}{4\Pi_0} \sum_{n>0} n \Pi_n J_n \right)$$

- achieved goal: Hamiltonian no longer commutes with everything!
Descendants are no longer soft

- replace again $i\{,\,\} \rightarrow [\,,\,]$
Descendants are no longer soft

- replace again $i\{,\} \rightarrow [ , ]$
- consider descendants

\[ J_{-n} |0\rangle \]

of highest weight vacuum $J_n |0\rangle = 0$ for all $n \geq 0$
Descendants are no longer soft

- replace again $i\{ , \} \to [ , ]$
- consider descendants

$$J_{-n} |0\rangle$$

of highest weight vacuum $J_n |0\rangle = 0$ for all $n \geq 0$
- calculate energy of such excitations

$$H_{\log} J_{-n} |0\rangle = [H_{\log}, J_{-n}] |0\rangle = \frac{\zeta \varepsilon}{J_0} n J_{-n} |0\rangle$$
Descendants are no longer soft

- replace again $i\{ , , \} \rightarrow [ , , ]$
- consider descendants

$$J_{-n}|0\rangle$$

of highest weight vacuum $J_{n}|0\rangle = 0$ for all $n \geq 0$
- calculate energy of such excitations

$$H_{\log}J_{-n}|0\rangle = [H_{\log}, J_{-n}]|0\rangle = \frac{\zeta \epsilon}{J_0} n J_{-n}|0\rangle$$

Energy eigenvalues linear in mode number $n$
Relations to fluff proposal? (Afshar, Grumiller, Sheikh-Jabbari, Yavartanoo '17)

- conjectured semi-classical set of BTZ microstates

\[
|\text{BTZ micro}(\{n_i^\pm\})\rangle = \prod_{J} J^+_{-n_i^+} J^-_{-n_i^-} |0\rangle
\]

labelled by positive integers \(\{n_i^\pm\}\) subject to spectral constraints

\[
\sum n_i^\pm = c \Delta^\pm \quad \Delta^\pm = \frac{1}{2} (\ell M_{\text{BTZ}} \pm J_{\text{BTZ}}) = \frac{c}{24} (J_0^\pm)^2
\]

- required input for fluff proposal: excitations fall into \(u(1)\) current algebra representations
- zero mode charge \(J_0\) has canonically conjugate \(\Phi_0\)
- soft hair degeneracy lifted to energies linear in mode number \(n\) all of the above fulfilled!

- missing piece of data: \(\zeta^\pm \epsilon = J^\pm_0 c\)

Fluff proposal intriguing, but not (yet) derived from first principles
Relations to fluff proposal? (Afshar, Grumiller, Sheikh-Jabbari, Yavartanoo '17)

- conjectured semi-classical set of BTZ microstates

\[ |\text{BTZ micro}\left(\{n_i^{\pm}\}\right)\rangle = \prod_{\pm} J_{-n_i^\pm}^\pm J_{-n_i^\pm}^- |0\rangle \]

labelled by positive integers \( \{n_i^{\pm}\} \) subject to spectral constraints

\[ \sum n_i^\pm = c \Delta^\pm \quad \Delta^\pm = \frac{1}{2} (\ell M_{\text{BTZ}} \pm J_{\text{BTZ}}) = \frac{c}{24} (J_0^\pm)^2 \]

- required input for fluff proposal:
  - excitations fall into \( u(1) \) current algebra representations
  - zero mode charge \( J_0 \) has canonically conjugate \( \Phi_0 \)
  - soft hair degeneracy lifted to energies linear in mode number \( n \)

all of the above fulfilled!
Relations to fluff proposal? (Afshar, Grumiller, Sheikh-Jabbari, Yavartanoo '17)

- conjectured semi-classical set of BTZ microstates

\[ |\text{BTZ micro}(\{n_i^\pm\})\rangle = \prod_{n_i^\pm} J_{n_i^\pm}^+ J_{n_i^\pm}^- |0\rangle \]

labelled by positive integers \(\{n_i^\pm\}\) subject to spectral constraints

\[ \sum n_i^\pm = c \Delta^\pm \quad \Delta^\pm = \frac{1}{2} (\ell M_{\text{BTZ}} \pm J_{\text{BTZ}}) = \frac{c}{24} (J_0^\pm)^2 \]

- required input for fluff proposal:
  - excitations fall into \(u(1)\) current algebra representations
  - zero mode charge \(J_0\) has canonically conjugate \(\Phi_0\)
  - soft hair degeneracy lifted to energies linear in mode number \(n\)

all of the above fulfilled!

- missing piece of data:

\[ \zeta^\pm_\epsilon = \frac{J_0^\pm}{c} \]
Relations to fluff proposal? (Afshar, Grumiller, Sheikh-Jabbari, Yavartanoo '17)

- conjectured semi-classical set of BTZ microstates

\[ |\text{BTZ micro}(\{n_i^\pm\})\rangle = \prod J^+_i J^-_i |0\rangle \]

labelled by positive integers \(\{n_i^\pm\}\) subject to spectral constraints

\[ \sum n_i^\pm = c \Delta^\pm \quad \Delta^\pm = \frac{1}{2} (\ell M_{\text{BTZ}} \pm J_{\text{BTZ}}) = \frac{c}{24} (J_0^\pm)^2 \]

- required input for fluff proposal:
  - excitations fall into \(u(1)\) current algebra representations
  - zero mode charge \(J_0\) has canonically conjugate \(\Phi_0\)
  - soft hair degeneracy lifted to energies linear in mode number \(n\)

all of the above fulfilled!

- missing piece of data:

\[ \zeta^\pm = \frac{J^\pm_0}{c} \]

Fluff proposal intriguing, but not (yet) derived from first principles
Relations to ultrarelativistic physics?

Carrollian limit

- Floreanini–Jackiw action

  has parameter $\mu$ giving the propagation speed of the chiral boson

  

  near horizon boundary action yields $\mu = 0$

  this is the Carrollian limit (compare with Donnay, Marteau and Penna)

  other consideration: start with bosonic string theory

  

  $X_{\mu}^{\pm}(t \pm \sigma) = x_{\mu}^{\pm} + \frac{\ell}{s} \sqrt{2} \sum_{n \neq 0} \alpha_{\pm} - n i e^{\pm i \sigma}$

  and take naive ultrarelativistic limit $t \to \epsilon t, \sigma \to \sigma, \epsilon \to 0$

  result

  $X_{\mu}^{\pm}(\sigma) = x_{\mu}^{\pm} + \frac{\ell}{s} \sqrt{2} \sum_{n \neq 0} \alpha_{\pm} - n i e^{\pm i \sigma}$

  equivalent to our on-shell mode expansion upon identifying $

  x_{\mu} = 2 \Phi_{0}^{\mu} \frac{\ell}{2} + \frac{\ell}{s} \alpha_{\pm} - n$

  $J_{n} - \text{sector comparison works analogously}$

  Confirms suspicion that nearly tensionless strings key in near horizon description of generic black holes
Relations to ultrarelativistic physics?

Carrollian limit

- Floreanini–Jackiw action

  has parameter $\mu$ giving the propagation speed of the chiral boson

- near horizon boundary action yields $\mu = 0$
Relations to ultrarelativistic physics?

Carrollian limit

- Floreanini–Jackiw action

  has parameter $\mu$ giving the propagation speed of the chiral boson

- near horizon boundary action yields $\mu = 0$

- this is the Carrollian limit (compare with Donnay, Marteau and Penna)
Relations to ultrarelativistic physics?

Ultrarelativistic strings

- other consideration: start with bosonic string theory

\[
X_\pm(t \pm \sigma) = \frac{x^\mu}{2} + \frac{\ell_s^2}{2} p_\pm(t \pm \sigma) + \frac{\ell_s}{\sqrt{2}} \sum_{n \neq 0} \frac{\alpha^\pm_n}{in} e^{in(t \pm \sigma)}
\]

and take naive ultrarelativistic limit \( t \rightarrow \epsilon t, \sigma \rightarrow \sigma, \epsilon \rightarrow 0 \)
Relations to ultrarelativistic physics?

Ultrarelativistic strings

- other consideration: start with bosonic string theory

\[ X_\mu^\pm(t \pm \sigma) = \frac{x^\mu}{2} + \frac{\ell_s^2}{2} p_\mu^\pm(t \pm \sigma) + \frac{\ell_s}{\sqrt{2}} \sum_{n \neq 0} \frac{\alpha_n^\pm}{in} e^{in(t \pm \sigma)} \]

and take naive ultrarelativistic limit \( t \to \epsilon t, \sigma \to \sigma, \epsilon \to 0 \)

- result

\[ X_\mu^\pm(\sigma) = \frac{x^\mu}{2} \pm \frac{\ell_s^2}{2} p_\mu^\pm \sigma + \frac{\ell_s}{\sqrt{2}} \sum_{n \neq 0} \frac{\alpha_{-n}^\pm}{in} e^{\pm in \sigma} \]

equivalent to our on-shell mode expansion upon identifying

\[ x^\mu = 2\Phi_0 \quad \ell_s^2 p_+^\mu = 2J_0 \quad \ell_s \alpha_{-n}^+ = \sqrt{2} J_n \]

- sector comparison works analogously
Relations to ultrarelativistic physics?

Ultrarelativistic strings

- other consideration: start with bosonic string theory

\[ X_{\pm}(t \pm \sigma) = \frac{x^\mu}{2} + \frac{\ell^2_s}{2} p^\mu_{\pm}(t \pm \sigma) + \frac{\ell_s}{\sqrt{2}} \sum_{n \neq 0}^{\infty} \frac{\alpha^\pm_{-n}}{in} e^{in(t \pm \sigma)} \]

and take naive ultrarelativistic limit \( t \to \epsilon t, \sigma \to \sigma, \epsilon \to 0 \)

- result

\[ X_{\pm}(\sigma) = \frac{x^\mu}{2} \pm \frac{\ell^2_s}{2} p^\mu_{\pm} \sigma + \frac{\ell_s}{\sqrt{2}} \sum_{n \neq 0}^{\infty} \frac{\alpha^\pm_{-n}}{in} e^{\pm in\sigma} \]

equivalent to our on-shell mode expansion upon identifying

\[ x^\mu = 2\Phi_0 \quad \ell^2_s p^\mu_+ = 2J_0 \quad \ell_s \alpha^+_n = \sqrt{2} J_n \]

- sector comparison works analogously

Confirms suspicion that nearly tensionless strings key in near horizon description of generic black holes
Thanks for your attention!