Topologically Massive Gravity –
A theory in 3 Dimensions

Sabine Ertl

Institute of Theoretical Physics
Vienna University of Technology
Lunch-Club

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Outline

- Motivation for 3 dimensional gravity
- Building the theory of Topologically Massive Gravity
- Finding solutions to TMG: Topologically Massive Mechanics
- Outlook
Why three dimensions?

Three-dimensional gravity is ...

... classically simpler

... physically interesting

... quantum mechanically solvable
Why three dimensions?

Three-dimensional gravity is ... 

... classically simpler

- Einsteins theory in 2+1 dimension probably the simplest model of gravity
- No local degree of freedom
- All perturbative solutions to Einsteins field equations are pure gauge
- Weyl tensor vanishes $\rightarrow$ Riemann tensor fully determined by Ricci tensor

... physically interesting

... quantum mechanically solvable
Why three dimensions?

Three-dimensional gravity is …

… classically simpler

… physically interesting

• Black holes (for $\Lambda < 0$): BTZ
  $\rightarrow$ Thermodynamics of BTZ black holes

• For a modified theory: gravitons

… quantum mechanically solvable
Why three dimensions?

Three-dimensional gravity is ...

... classically simpler

... physically interesting

... quantum mechanically solvable

- Promising toy model to approach the quantization of GR
- Toy model for quantum gravity
- AdS/CFT correspondence
Building the theory: TMG

Let’s start with Einstein-Gravity in 3 dimensions

\[ l_{EH} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \, R \]

equations of motion:

\[ R_{\mu\nu} = 0 \]

no black holes!
Building the theory: TMG

Let’s add a cosmological constant:

\[ I_{\text{EH+CC}} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} (R - 2\Lambda) \]

Equations of motion:

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \]

- 3 vacuum solutions: Minkowski ($\Lambda = 0$), de-Sitter ($\Lambda > 0$), anti de-Sitter ($\Lambda < 0$)
- For $\Lambda < 0$, use parameterization $\Lambda = -\frac{1}{\ell^2} \rightarrow R = 6\Lambda$
- BTZ black hole

But still no graviton!
Building the theory: TMG

Let’s add a gravitational Chern-Simons term

\[ I_{\text{TMG}} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[ R + \frac{2}{\ell^2} + \frac{1}{2\mu} \varepsilon^{\alpha\beta\gamma} \Gamma_{\alpha\sigma}^\rho \left( \partial_\beta \Gamma_{\gamma\rho}^\sigma + \frac{2}{3} \Gamma_{\beta\tau}^\sigma \Gamma_{\gamma\rho}^\tau \right) \right] \]

- Massive propagating degree of freedom and 2 massless boundary gravitons [Li, Song, Strominger [1]]
- CS-Term: maximally chiral
- Equations of motion

\[ G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0 \]

Black holes and Gravitons!
Some Aspects of TMG

\[ I_{\text{TMG}} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[ R + \frac{2}{\ell^2} + \frac{1}{2\mu} \varepsilon^{\alpha\beta\gamma} \Gamma_{\alpha\sigma}^\beta (\partial_\beta \Gamma_{\gamma\rho}^\sigma + \frac{2}{3} \Gamma_{\beta\tau}^\sigma \Gamma_{\gamma\rho}^\tau) \right] \]

- Unstable/inconsistent for generic \( \mu \rightarrow \) choice of sign of EH-term
- Exception: chiral/logarithmic point \( \mu \ell = 1 \):
  2 different theories exist (depending on boundary conditions):
  - Brown-Henneaux: chiral CFT (unitary) [Li, Song, Strominger [1]]
  - Grumiller-Johansson: LCFT (non-unitary) [Grumiller, Johansson [3]]
- Additional boundary terms [Kraus, Larsen [2]]

\[ I_{\text{GHY} + \text{BCC}} = \frac{1}{8\pi G} \int d^2x \sqrt{-g} \left( K - \frac{1}{\ell^2} \right) \]
Topologically Massive Mechanics

Finding solutions to TMG is tricky (trivial solutions, no-go theorems), therefore

**Simplification: stationary axi-symmetric TMG**

[Clement 1994 [4] and in 2010 Ertl, Grumiller, Johansson [5]]

Set up: stationary axi-symmetric 3d lineelement + 2d metric → $I_{TMG}$

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \frac{e^2}{X^2} d\rho^2 \]

\[ g_{\mu\nu} = \begin{pmatrix} X^+ & Y \\ Y & X^- \end{pmatrix} \]

with \( X = (X^+, X^-, Y) \)

and \( \det g = X^2 = X^i X_i = X^i X^j \eta_{ij} = X^+ X^- - Y^2 \)

\( X^2 = 0 : 'centre' \)
Finding solutions to TMG is tricky (trivial solutions, no-go theorems), therefore

**Simplification:** stationary axi-symmetric TMG

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\[
I_{TMM} = \int d\rho \ e \left( \frac{1}{2} e^{-2} \dot{\mathbf{X}}^2 - \frac{2}{\ell^2} - \frac{1}{2\mu} e^{-3} \epsilon_{ijk} \dot{X}^i \dot{X}^j \ddot{X}^k \right)
\]

Hamiltonian constraint: \( G = \frac{1}{2} \dot{\mathbf{X}}^2 + \frac{2}{\ell^2} - \frac{1}{\mu} \epsilon_{ijk} X^i \dot{X}^j \ddot{X}^k = 0 \)

Equations of motion: \( \ddot{X}_i = -\frac{1}{2\mu} \epsilon_{ijk} (3 \dot{X}^j \ddot{X}^k + 2X^j \dot{X}^k) \)
Topologically Massive Mechanics

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Conserved angular momentum (first integrals to equations of motion)

\[ J_i = \epsilon_{ijk} X^j \dot{X}^k - \frac{1}{4\mu} \left( 5\dot{X}^2 + \frac{12}{\ell^2} \right) X_i + \frac{1}{2\mu} (X\dot{X}) \dot{X}_i - \frac{1}{\mu} X^2 \ddot{X}_i \]

combination with Hamiltonian constraint

\[ \epsilon_{ijk} J^i X^j \dot{X}^k = \frac{1}{2} X^2 \dot{X}^2 - (X\dot{X})^2 - \frac{2}{\ell^2} X^2 \]

using equations of motion

\[ XJ = \frac{1}{2\mu} \left( (X\dot{X})^2 - X^2 \dot{X}^2 \right) \]
Finding solutions to TMG is tricky (trivial solutions, no-go theorems), therefore

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simple but difficult to find analytic solutions → non-existence results:
• $|\mu l| = 1$: Einstein solutions
• $|\mu l| = 3$: null warped black hole
Classification of all Solutions

In general: 6d phase space containing a 4d subspace (Einstein, Schrödinger, Warped AdS) \(\rightarrow\) classification into 4 sectors:

**Einstein** \(\dddot{X} = 0\)

**Schrödinger** \(\dddot{X} \neq 0\), linear dependence of \(X, \dot{X}, \ddot{X}\)

**Warped** \(\dddot{X}^2 = \dddot{X} = 0\), linear independence of \(X, \dot{X}, \ddot{X}\)

**Generic** \(\dddot{X}^2 \neq 0\) and/or \(\dot{X}\dddot{X} \neq 0\)
Classification of all Solutions

Einstein $\ddot{X} = 0$

- Solution to EOM: $X = X_{(0)} \rho + X_{(2)}$
- All stationary, axisymmetric solutions of Einstein gravity
- Solutions are locally and asymptotically adS

$$X = \left( a, \frac{1}{a}, \pm \frac{2}{\ell} \rho + 1 \right) \quad X = \left( a, 0, \pm \frac{2}{\ell} \rho \right)$$

Schrödinger $\ddot{X} \neq 0$, linear dependence of $X, \dot{X}, \ddot{X}$

Warped $\dddot{X} = \ddot{X} = 0$, linear independence of $X, \dot{X}, \ddot{X}$

Generic $\dddot{X} \neq 0$ and/or $\dot{X} \ddot{X} \neq 0$
Classification of all Solutions

**Einstein** \( \ddot{X} = 0 \)

BTZ solution

\[ ds^2 = a (dx^+)^2 + \frac{1}{a} (dx^-)^2 \pm \left( e^{2r} \frac{1}{\alpha} + e^{-2r} \alpha \right) dx^+ dx^- - \ell^2 dr^2 \]

**Schrödinger** \( \ddot{X} \neq 0 \), linear dependence of \( X, \dot{X}, \ddot{X} \)

**Warped** \( \dddot{X} = \ddot{X} = 0 \), linear independence of \( X, \dot{X}, \ddot{X} \)

**Generic** \( \dddot{X} \neq 0 \) and/or \( \dot{X} \ddot{X} \neq 0 \)
Classification of all Solutions

Einstein  \( \ddot{X} = 0 \)

BTZ solution

\[ ds^2 = -4G\ell(L\, du^2 + \bar{L}\, dv^2) - (\ell^2 e^{2r} + 16G^2 L\bar{L}e^{-2r})\, du\, dv - \ell^2\, dr^2 \]

\[ L = \frac{(r_+ + r_-)^2}{16G\ell} \quad \bar{L} = \frac{(r_+ - r_-)^2}{16G\ell} \quad m = L + \bar{L} \quad j = L - \bar{L} \]

Schrödinger  \( \ddot{X} \neq 0 \), linear dependence of \( X, \dot{X}, \ddot{X} \)

Warped  \( \ddot{X}^2 = \dddot{X} = 0 \), linear independence of \( X, \dot{X}, \ddot{X} \)

Generic  \( \ddot{X}^2 \neq 0 \) and/or \( \dot{X}\ddot{X} \neq 0 \)
Classification of all Solutions

Einstein \( \ddot{X} = 0 \)

Schrödinger \( \ddot{X} \neq 0 \), linear dependence of \( X, \dot{X}, \ddot{X} \)

- Solutions are: \( X = \left( s\rho^{(1\mp\mu\ell)/2} + a\rho + b, \ 0, \ \pm \frac{2}{\ell} \rho \right) \)
- Spacetimes with asymptotic Schrödinger behaviour:
  \[
  \left. ds^2 \right|_{r \to 0} \sim \ell^2 \left( \frac{\pm 2 \ dx^+ \ dx^- - dr^2}{r^2} + \beta \frac{(dx^+)^2}{r^{2z}} \right) \quad z = \frac{1 \mp \mu \ell}{2} \]
  \[
  \mu \ell = \mp 3: \ z = 2: \text{null warped AdS} \]

Warped \( \dot{X}^2 = \dddot{X} = 0 \), linear independence of \( X, \dot{X}, \dddot{X} \)

Generic \( \dddot{X} \neq 0 \) and/or \( \dot{X} \dddot{X} \neq 0 \)
Classification of all Solutions

Einstein \( \ddot{X} = 0 \)

Schrödinger \( \ddot{X} \neq 0 \), linear dependence of \( X, \dot{X}, \ddot{X} \)

\[
\begin{align*}
    ds^2 &= \left(s\rho^{(1+\mu\ell)/2} + a\rho + b \right)(dx^+)^2 \pm \frac{4\rho}{\ell} \, dx^+ \, dx^- - \frac{\ell^2 \, d\rho^2}{4\rho^2} \\
\end{align*}
\]

- \( z < 1 \): Asymptotic AdS
- \( z = 1; \mu\ell = 1 \): 2 logarithmic solutions
  - Asymptotically AdS: reminiscent of the log-mode [Grumiller, Johansson [3]]
  - Marginally violated asymptotically AdS condition [Skenderis et. all.[6]] [Grumiler, Sachs [7]]

Warped \( \ddot{X}^2 = \dddot{X} = 0 \), linear independence of \( X, \dot{X}, \ddot{X} \)

Generic \( \ddot{X}^2 \neq 0 \) and/or \( \dot{X}\dddot{X} \neq 0 \)
Classification of all Solutions

Einstein \( \ddot{X} = 0 \)

Schrödinger \( \ddot{X} \neq 0 \), linear dependence of \( X, \dot{X}, \ddot{X} \)

Warped \( \dddot{X} = \ddot{X} = 0 \), linear independence of \( X, \dot{X}, \ddot{X} \)

- General solutions \( X = X_{(-2)} \rho^2 + X_{(0)} \rho + X_{(2)} \)
- Locally and asymptotically warped (squashed or stretched) AdS
  [Nutku [8]] [Anninos, Li, Padi, Song, Strominger [9]]
- Warped AdS candidate for stable TMG backgrounds

Generic \( \dddot{X} \neq 0 \) and/or \( \dot{X} \dddot{X} \neq 0 \)
Classification of all Solutions

Einstein \( \ddot{X} = 0 \)

Schrödinger \( \ddot{X} \neq 0 \), linear dependence of \( X, \dot{X}, \ddot{X} \)

Warped \( \dddot{X} = \dddot{X} = 0 \), linear independence of \( X, \dot{X}, \ddot{X} \)

Generic \( \dddot{X} \neq 0 \) and/or \( \dddot{X} \dddot{X} \neq 0 \)

These solutions are neither Einstein, Schrödinger nor warped AdS
In fact: Any generic solution must be non-polynomial in \( \rho \)
The generic sector is described by the constraints: $\dddot{X}^2 \neq 0$ and/or $\ddot{X}\dot{X} \neq 0$

Solving for solutions: numerical analysis
Example of the Generic Center I

Naked Singularity – non-analytic center

![Graphs showing various functions and their derivatives against the variable ρ.](image-url)
Example of the Generic Center II

Soliton - no center
Zooming out ...

... evidence for asymptotic warped AdS behaviour
... damped oscillations around warped AdS
Outlook

• a lot of new 3D gravity theories: NMG, GMG, MSG, HOMG, BIG
• apply TMM recipe on those novel theories
• CFT’s
• create new 3D theories

• still open questions in TMM
  • Topography of landscape of solutions
  • boundary conditions and corresponding asymptotic symmetry group
  • stability?
  • Soliton interpretation as finite energy excitations around WAdS?
  • Soliton asymptotics to AdS or Schrödinger?
  • Kink solutions?
Thank you for your attention
References


