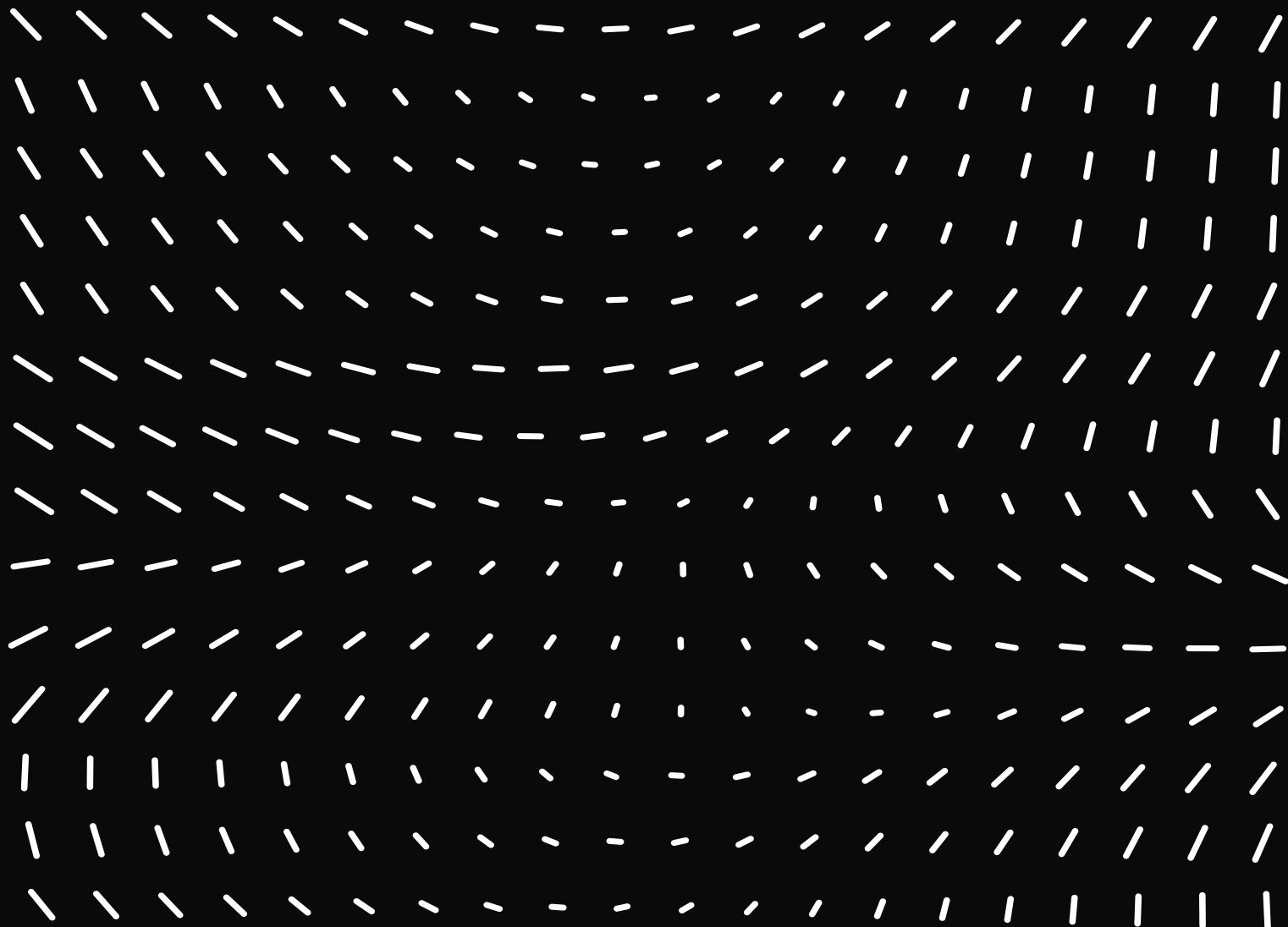


Chaos

in 3D HS Gravity

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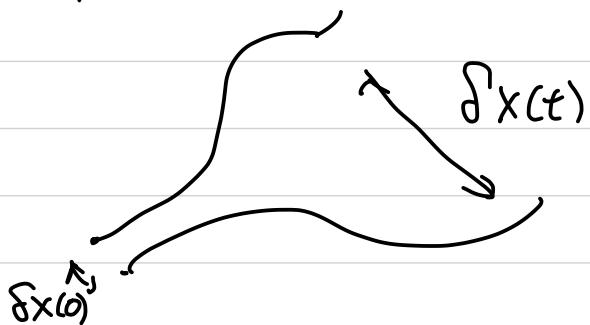
Chaos in 3D Higher Spin Gravity

Based on 1903. 08761 Prithvi Narayan, JY

(1903 . 09086 Viktor Johke, Keun-Young Kim, JY)
↳ rotating BTZ Black hole.

I. Introduction.

* Butterfly effect.



◦ Classical.

$$\frac{\delta x(t)}{\delta x(0)} \sim e^{\lambda t}.$$

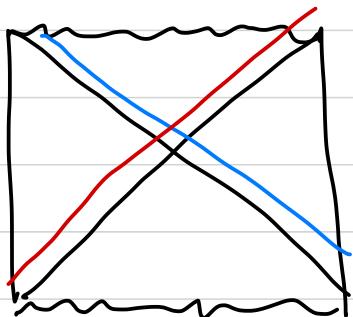
◦ Quantum $\langle [V(t), W(0)]^2 \rangle_B \sim e^{\lambda t}$

◦ Out-of-time-ordered correlator.

$$\Rightarrow \langle V(t) W(0) V(0) W(0) \rangle_B \sim 1 - \epsilon e^{\lambda t}.$$

* Bound on chaos. : $\lambda_L \leq \frac{2\pi}{\beta}$.
Maldacena, Shenker, Stanford.

* Black hole : maximally chaotic



High energy scattering of
two shock wave!

: exchange of graviton.

$$\sim e^{\frac{2\pi}{\beta} t}$$

: saturate!

Spin- s field exchange?



$$\Rightarrow e^{\frac{2\pi}{\beta}(s-1)t}$$

$$\lambda_s = \frac{2\pi}{\beta}(s-1) : \text{violate chaos bound}$$

// for $s > 2$.

HS gravity.

i) HS symmetry is broken.

ii) HS symmetry is not broken., infinite tower of HS field.

$$\langle OTOC \rangle \sim 1 + e^{\frac{2\pi}{\beta}t} + e^{\frac{4\pi}{\beta}t} + \dots$$

$$= \frac{1}{1 - e^{\frac{2\pi}{\beta}t}} \cdot \lambda = 0 ..$$

iii) Finite tower of HS field.

3D SL(N) CS gravity. $s=2, 3, \dots, N$.

: semi-classical limit of $\lambda_s[\lambda]$ HS gravity.

$$(\lambda \rightarrow -N) \Rightarrow \text{formation}$$

non-unitary. Causality in 4 pt function ??

E. Perlmutter : W vacuum block.

Motivation:

How to calculate spin-s field contribution
to 4pt function in $SL(N,\mathbb{C})$ CS
gravity?



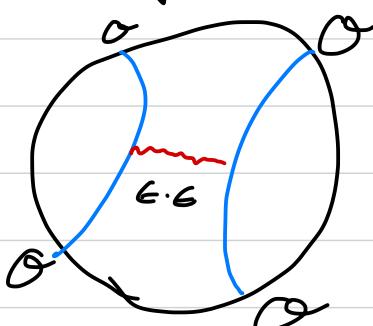
cf) JT gravity / Schwarzian theory / SYK model

$$S = \int dt \cdot Sch \left[\tan \frac{\pi f(t)}{\beta}, t \right] \quad f: \text{boundary gravitor}$$

$$= \int dt \cdot Sch[f(t), t] + \frac{2\pi^2}{\beta^2} [f'(t)]^2 \quad (\text{Pseudo Goldstone boson})$$

SSB/ESB of
reparametrization

Codimension orbit action, $Diff(S)/SL(2)$.



$$Sch[f(t), t] = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2$$

① Quadratic level : $f(t) = t + \epsilon e^{i\omega t} - \frac{2\pi i n}{\beta} t$.
& mode expansion : $= t + \sum_n \epsilon_n e^{-\frac{2\pi i n}{\beta} t}$.

$$S^{(2)} = \pi^2 (n^2 - 1) \epsilon_n \epsilon_{-n}$$

: vanishes at $n=0, \pm 1$

$$\textcircled{2} \quad G(t_1, t_2) = \langle O(t_1) O(t_2) \rangle \sim \frac{1}{(\sin \frac{\pi(t_2-t_1)}{\beta})^2}$$

$$\xrightarrow{(f(t_1))} G(f(t_1), f(t_2)) \xleftarrow{(f'(t_2))} = G(t_1, t_2) + \epsilon_n \delta_{nn} G(t_1, t_1)$$

$f(t) = t + \epsilon_n e^{-\frac{2\pi i n}{\beta} t} + \dots$

③

$$\boxed{\langle G_n G_{-n} \rangle S_{Gn} G(t_1, t_2) S_{G-n} G(t_3, t_4)}.$$

Questions.

I. What is W-generalization of Schwarzian action?

Goguri, Bershadsky (1989) & Wei Li, Theisen (2015)
Grumiller et al. (2018)

① How to get the on-shell action?
(in CS gravity)
- Boundary term, Boundary condition.

② finite temperature Schwarzian?

(smooth gauge trans vs Large gauge trans)

$$Sch[f(t), t] \implies Sch\left[\tan \frac{\pi f(t)}{\beta}, t\right]$$

$$f: R \rightarrow R$$

$$f(t): S' \rightarrow S'$$

II. (boundary- \mathbb{R}^3 -boundary) two point function.

① What kind of two point function.

pure CS + no matter
topological : difficult to couple to matter.

② W_S -transformation of two-point function

③ Why do we calculate
 $\langle G_n G_{-n} \rangle S_{G_n} G_1 S_{G_{-n}} G_2 ?$

I. \mathcal{W} -generalized Schwarzian :

$$T = - \left\{ \frac{\partial^3 e}{\partial e} - \frac{4}{3} \left(\frac{\partial^2 e}{\partial e} \right)^2 \right\} + \frac{1}{6} \frac{\partial^2 e}{\partial e} \frac{\partial^2 f}{\partial f} + (e \leftrightarrow f) \quad (3.78)$$

$$W = - \frac{1}{6} \left\{ \frac{\partial^4 e}{\partial e} - 5 \frac{\partial^3 e}{\partial e} \frac{\partial^2 e}{\partial e} + \frac{40}{9} \left(\frac{\partial^2 e}{\partial e} \right)^3 \right\} + \frac{1}{6} \frac{\partial^3 e}{\partial e} \frac{\partial^2 f}{\partial f} + \frac{5}{18} \frac{\partial^2 e}{\partial e} \left(\frac{\partial^2 f}{\partial f} \right)^2 - (e \leftrightarrow f)$$

* Quick derivation:

$$A = b \bar{c}(d + \alpha z) b c : CS \text{ connection}$$

$$b = e^{r L_0},$$

$$\Rightarrow \alpha_z = g_{(z)}^{-1} \partial_z g^{(z)} \& \text{Gauss Decomposition } g = L D U \\ = \begin{pmatrix} 0 & L & N \\ I & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix}$$

① SLC(N,C) CS gravity

$$A, \bar{A}.$$

$$I_{CS} = \frac{i k_{CS}}{4\pi} \int_M \text{Tr} \left(A dA + \frac{2}{3} A^3 \right) - \dots$$

$$S I_{CS} \sim \int_M \text{Tr}(A \delta A) \quad r, z, \bar{z}$$

$$= \int_M \text{Tr} (A_z \delta A_{\bar{z}} - A_{\bar{z}} \delta A_z)$$

$$A_{\bar{z}} = 0 : \text{good variational principle.}$$

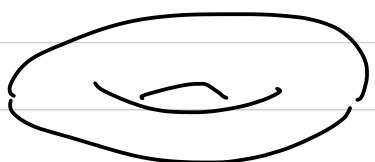
Spin-3 charged black hole: X
(chemical potential.)

$$A = b^{-1} (d + \alpha(z)) b \quad b = e^{r L_0}.$$

Asymptotic AdS condition : $[L_n, L_m] = (n-m)L_{n+m}$.
 $A - A_{\text{AdS}} \sim O(1)$

$$\Rightarrow \alpha(z) = L_1 - \frac{2\pi}{h} L(z) L_{-1} + \frac{\pi}{2h} W(z) k_{-2}$$

Jan de Boer, Juan Jottar . 1302.0816.



Solid torus,

$$z \sim z + 2\pi \sim z + \tau ..$$

In variation, consider δz : periodicity can be changed

$$\omega \sim \omega + 2\pi \sim \omega + 2\pi i ..$$

$$d\omega \wedge d\bar{\omega} = \frac{4\pi dz}{\text{Im}(z)} .. \quad d\delta^2 = dz d\bar{z} =$$

Upshot :

$$\Downarrow \text{keep volume } d\omega \wedge d\bar{\omega} = \frac{4\pi dz}{\text{Im}(z)},$$

\Rightarrow Add boundary term

$$I_b = -\frac{k_{cs}}{2\pi} \int_{\partial M} d^2 z \text{Tr} ((a_z - 2L_1)a_{\bar{z}}) - \frac{k_{cs}}{2\pi} \int_{\partial M} d^2 z \text{Tr} ((\bar{a}_{\bar{z}} - 2L_{-1})\bar{a}_z),$$

\Rightarrow Variational principle .

$$\begin{aligned} \delta I_{\text{tot}} = & -ik_{cs} \int_{\partial M} \frac{d^2 z}{2\pi \text{Im}(\tau)} \text{Tr} \left[(a_z - L_1)\delta((\bar{\tau} - \tau)a_{\bar{z}}) + \left(\frac{a_z^2}{2} + a_z a_{\bar{z}} - \frac{\bar{a}_{\bar{z}}^2}{2}\right) \delta\tau \right. \\ & \left. - (-\bar{a}_{\bar{z}} - L_{-1})\delta((\bar{\tau} - \tau)\bar{a}_z) + \left(\frac{\bar{a}_{\bar{z}}^2}{2} + \bar{a}_{\bar{z}} \bar{a}_z - \frac{a_z^2}{2}\right) \delta\bar{\tau} \right] \end{aligned}$$

$$\alpha_{\bar{z}} = 0, \quad S_{\bar{z}} = 0.$$

$$\Rightarrow \text{Ion-shell} = \frac{i k c s}{2\pi} \int \frac{dz^2}{\text{Im}(z)} \text{Tr} \left[\frac{c}{2} \alpha_z^2 - \frac{\bar{c}}{2} \bar{\alpha}_{\bar{z}}^2 \right]$$

$$= i \int \frac{dz^2}{\text{Im}(z)} (c S(z) - \bar{c} S(\bar{z}))$$

$$\alpha_z = \begin{pmatrix} 0 & \#L & \#W \\ \Sigma & 0 & \#L \\ 0 & \Sigma & 0 \end{pmatrix}$$

* No spin-3 chemical potential.

\Rightarrow No $W(z)$ in action : future work.

\times L_0 : const. solution \Rightarrow BTZ, ($L_0 > 0$)
 $\Rightarrow \alpha_{\text{BTZ}} = L_1 - f_0 L_{-1}$
 $\alpha_z = h(z)(\alpha_{\text{BTZ}} + \partial_z)h(z)$.

h : smooth residual gauge transf
 \hookrightarrow does not change holonomy / can be connected to identity.

* Holonomy along Euclidean time direction

$$H_0(A) = b^\dagger h^{-1} e^{-w} h b \quad \underbrace{\qquad}_{\text{contractible cycle (along time)}} \quad \text{Euclidean BH.}$$

Smoothness condition



$$w = \tau \alpha_z + \bar{\tau} \alpha_{\bar{z}} \sim 2\pi i L_0.$$

$$\Rightarrow \tau = i \sqrt{\frac{k c s}{2\pi L_0}},$$

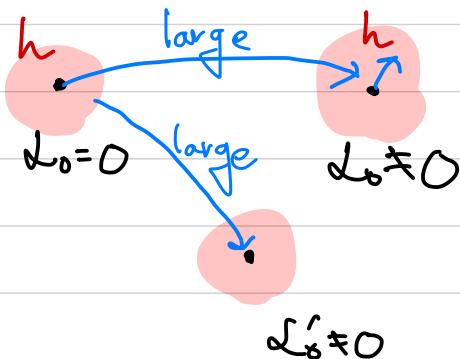
Change holonomy \Rightarrow change τ .

$\Rightarrow \delta\tau = 0$ implies \mathcal{L}_0 : fixed (around fixed background.)

② finite temperature Schwarzsian?

* Difficult to parameterize large & smooth gauge transf separately.

\Rightarrow Give up the finite transf.



(smooth fluctuation around stationary background)

* Consider fixed $\overset{\sim}{\mathcal{L}}_0$ \rightarrow infinitesimal residual gauge (stationary background) transf.,

: guarantee smooth residual gauge transf.

$$\Rightarrow h = 1 + \epsilon \lambda^{(1)} + \frac{1}{2} \epsilon^2 \left[(\lambda^{(1)})^2 + \lambda^{(2)} \right].$$

$\lambda^{(1)}, \lambda^{(2)}$: traceless 3×3 matrix

8 d.o.f 8 d.o.f

1st order: gauge fixing condition $\Rightarrow \eta, \zeta$

2nd order: $\therefore \Rightarrow \eta^2, \zeta^2, \eta\zeta$

6 eqs.

$f^{(1)}, g^{(2)}$

2 parameters.

$$I_{\text{on-shell}} = I_0 + \epsilon \underbrace{I^{(1)}}_{\text{total derivative}} + \epsilon^2 I^{(2)} + \dots$$

$$\epsilon^2 I_{\text{on-shell}}^{(2)} = \frac{4\pi^4 i k_{cs}}{\tau^3} \sum_{n \geq 2} n^2(n^2 - 1) \eta_{-n} \eta_n + \frac{2\pi^6 i k_{cs}}{3\tau^5} \sum_{n \geq 3} n^2(n^2 - 1)(n^2 - 4) \zeta_{-n} \zeta_n$$

\nearrow
Vanishes at $n=0, \pm 1$,
graviton

\nearrow
Vanish at $n=0, \pm 1, \pm 2$.
spin 3 field.

II. Vasiliev equation ($SL(N)$)

$$dC + AC - C \bar{A} = 0.$$

* For stationary background

$$C(r, z, \bar{z}) = b(r) e^{-\alpha_{BTZ} z} c_0 e^{\bar{\alpha}_{BTZ} \bar{z}} b^{-1}(r)$$

* For non-constant A .

$$C = \lim_{r \rightarrow \infty} e^{2hr} e^{-\int_{r,z}^{r,z'} A} c_0 e^{-\int_{r,z}^{r,z'} \bar{A}}$$

$$h = -\frac{N-1}{2} < 0 \text{ (semiclassical)}$$

$$\Phi(r, z, \bar{z}; z', \bar{z}')$$

$$= \text{tr}(C) = \text{tr}\left(\lim_{r \rightarrow \infty} e^{2hr} e^{-\int_Q^P A} c_0 e^{-\int_Q^P \bar{A}}\right)$$

: Wilson line.

$$\text{Tr} \left(b^{-1}(r_1) h^{-1}(z_1) e^{-a_{BTZ}(z_1 - z_2)} h(z_2) c_0 \bar{h}(\bar{z}_2) e^{-\bar{a}_{BTZ}(\bar{z}_2 - \bar{z}_1)} \bar{h}^{-1}(\bar{z}_1) b^{-1}(r_1) \right)$$

\vdots

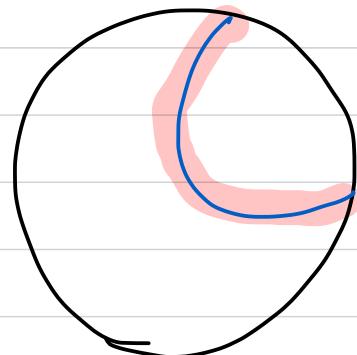
$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \end{pmatrix} \text{ "highest weight"}$$

$$\star \Phi(z_1, \bar{z}_1, z_2, \bar{z}_2) = \lim_{r_i \rightarrow \infty} e^{2\pi r_i} \Phi^*(r_i, z_1, \bar{z}_1, z_2, \bar{z}_2)$$

: boundary-to-boundary propagator.
 \Rightarrow geodesic distance.

* For Non-constant background

\Rightarrow gravitationally dressed
 Wilson line (master field)
 by boundary graviton
 (soft)



* Soft mode expansion

$$h = 1 + n(z) \quad \xrightarrow{n_n e^{-\frac{2\pi i n}{\tau} z}} \text{(mode expansion)}$$

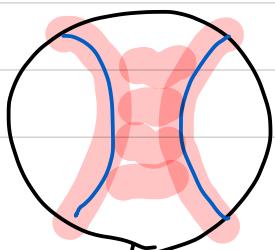
$$\Rightarrow \Phi_{(z_1, z_2)}^{\text{dressed}} = \Phi^*(z_1, z_2) + n_m s_n \Phi_{(z_1, z_2)}^{\text{dressed}} + \dots$$

$$G(z_1, z_2)$$

: two-point function
 in constant background

$$\langle \Phi_{(z_1, z_2)}^{\text{dressed}} \Phi_{(z_3, z_4)}^{\text{dressed}} \rangle$$

$$= G(1,2)G(3,4) + \langle n_m, n_{-n} \rangle s_m \Phi_{(1,2)}^* \int n_m \Phi_{(3,4)}^*$$



$$= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

* Soft mode eigenfunction

$$f_{2,n} \equiv \gamma_2 e^{-\frac{2\pi i n \chi}{\tau}} \left[n \cos \frac{2\pi n \sigma}{\tau} - \frac{\sin \frac{2\pi n \sigma}{\tau}}{\tan \frac{2\pi \sigma}{\tau}} \right] = \frac{S_{n,\omega} \Phi_{dressed}(1,2)}{G_1(1,2)}$$

$$f_{3,n} \equiv \gamma_3 e^{-\frac{2\pi i n \chi}{\tau}} \left[2n^2 \sin \frac{2\pi n \sigma}{\tau} + 6n \frac{\cos \frac{2\pi n \sigma}{\tau}}{\tan \frac{2\pi \sigma}{\tau}} - 2 \frac{1 + 2 \cos^2 \frac{2\pi \sigma}{\tau}}{\sin^2 \frac{2\pi \sigma}{\tau}} \sin \frac{2\pi n \sigma}{\tau} \right] = \frac{S_{n,\omega} \Phi_{dressed}(1,2)}{G_1(1,2)}$$

$$\chi = \frac{1}{2}(z_1 + z_2), \quad \sigma = \frac{1}{2}(z_1 - z_2)$$

classical

quantum fluctuations

$$(2) \cdot \Phi_{dressed} = G_1 + n_n S_{n,\omega} \Phi_{dressed} + \dots$$

$$\langle n_n \Phi_{dressed} \rangle \sim S_{n,\omega} \Phi_{dressed} + \dots$$

Ward identity

* W-transf of two point function?

At least, a special operator.

Observation: null relation in W_{KK} minimal model
in Large C.

(□ : ϕ) (dual to scalar field)

$$W_0^{(3)} \phi_1 = w^{(3)} \phi_1 \quad w^{(3)} = -(1 + \lambda)(2 + \lambda)$$

$$W_{-1}^{(3)} \phi_1 = \frac{3w^{(3)}}{2h} L_{-1} \phi_1$$

$$W_{-2}^{(3)} \phi_1 = \frac{3w^{(3)}}{h(2h + 1)} L_{-1}^2 \phi_1$$

$$\oint \langle \phi_1(z_1) \phi_1(z_2) \rangle = -\frac{1}{2\pi i} \oint dz \zeta(z) \langle W^{(3)}(z) \phi_1(z) \phi_1(z_2) \rangle$$

= ...

$L_i \rightarrow \partial_{z_i}$

$$\zeta \rightarrow e^{-\frac{2\pi i \beta}{c}} \text{ for finite temperature.}$$

$$S_n \langle \phi, \phi \rangle = G_{(1,2)} f_{3,n}(x, \sigma)$$

* Conjecture. for null relations.

$$W^{(s)}_{-n} \phi \sim (L_{-1})^n \phi$$

then, we can fix coefficient.

$$\Rightarrow S_{\zeta^{(s)}} \langle \quad \rangle = \text{Hypergeometric functions.}$$

For even s , one can simplify them as

$$f_{s,n}(\chi, \sigma) = 2\sqrt{\pi} e^{-in\chi} (n+s-1)! (-1)^s \frac{(\sin \sigma)^s}{s^n} \\ \times \left[\cos \frac{\pi i n}{2} \frac{2F_1(\frac{s+n}{2}, \frac{s-n}{2}, \frac{1}{2}, \cos^2 \sigma)}{\Gamma(\frac{n+s+1}{2}) \Gamma(\frac{n-s+2}{2})} + 2 \sin \frac{\pi i n}{2} \cos z \frac{2F_1(\frac{s+n+1}{2}, \frac{s-n+1}{2}, \frac{3}{2}, \cos^2 \sigma)}{\Gamma(\frac{n+s}{2}) \Gamma(\frac{n-s+1}{2})} \right] \quad (B.17)$$

For odds, we have

$$f_{s,n}(\chi, \sigma) = 2\sqrt{\pi} e^{-in\chi} (n+s-1)! (-1)^s \frac{(\sin \sigma)^s}{s^n} \\ \times \left[\sin \frac{\pi i n}{2} \frac{2F_1(\frac{s+n}{2}, \frac{s-n}{2}, \frac{1}{2}, \cos^2 \sigma)}{\Gamma(\frac{n+s+1}{2}) \Gamma(\frac{n-s+2}{2})} - 2 \cos \frac{\pi i n}{2} \cos \sigma \frac{2F_1(\frac{s+n+1}{2}, \frac{s-n+1}{2}, \frac{3}{2}, \cos^2 \sigma)}{\Gamma(\frac{n+s}{2}) \Gamma(\frac{n-s+1}{2})} \right] \quad (B.18)$$

appears in SYK model

as basis in limit $g \rightarrow \infty$, $v \rightarrow 1$,

They are solution of

$$\left[n^2 + \partial_\sigma^2 - \frac{s(s-1)}{\sin^2 \sigma} \right] G_{s,n}(\sigma) = 0$$

* Recursion relation.

We know $f_{1,n}(x, \sigma) \sim \frac{\sin n\sigma}{\sin \sigma}$ $f_{2,n}(x, \sigma) \sim \frac{1}{\sin \sigma} (n \cos n\sigma - \frac{\sin n\sigma}{\tan \sigma})$

↑ from (U) transf ↑ from conformal transf.
($s=2$)

* Observation : $f_{2,n}(x, \sigma) = \partial_x f_{1,n}(x, \sigma)$

* Can we construct $f_{s,n}(x, \sigma)$ out of $f_{s,n}'$?

$f_{s,n} = e^{-in\sigma} g_{s,n}(\sigma)$ s.t. $\langle f_{s,n}, f_{s',n'} \rangle \sim f_{s,s'} f_{n,n'}$
 trans inv of center of coordinates ($sL(s) + sL(s')$)

* We found that

$$g_{s+2,n}(\sigma) = \theta g_{s+1,n}(\sigma) + \frac{s^2(a^2-s^2)}{4s^2-1} g_{s,n}(\sigma)$$

Then, we have.

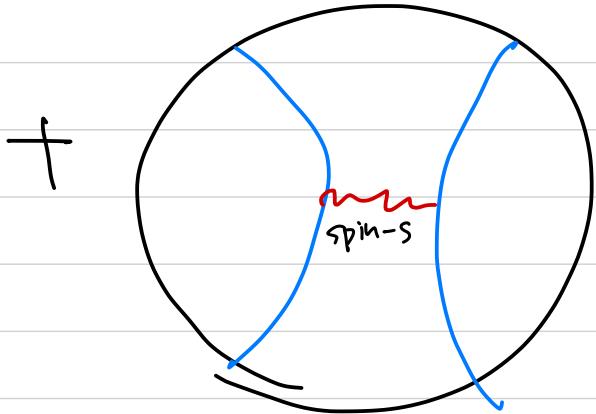
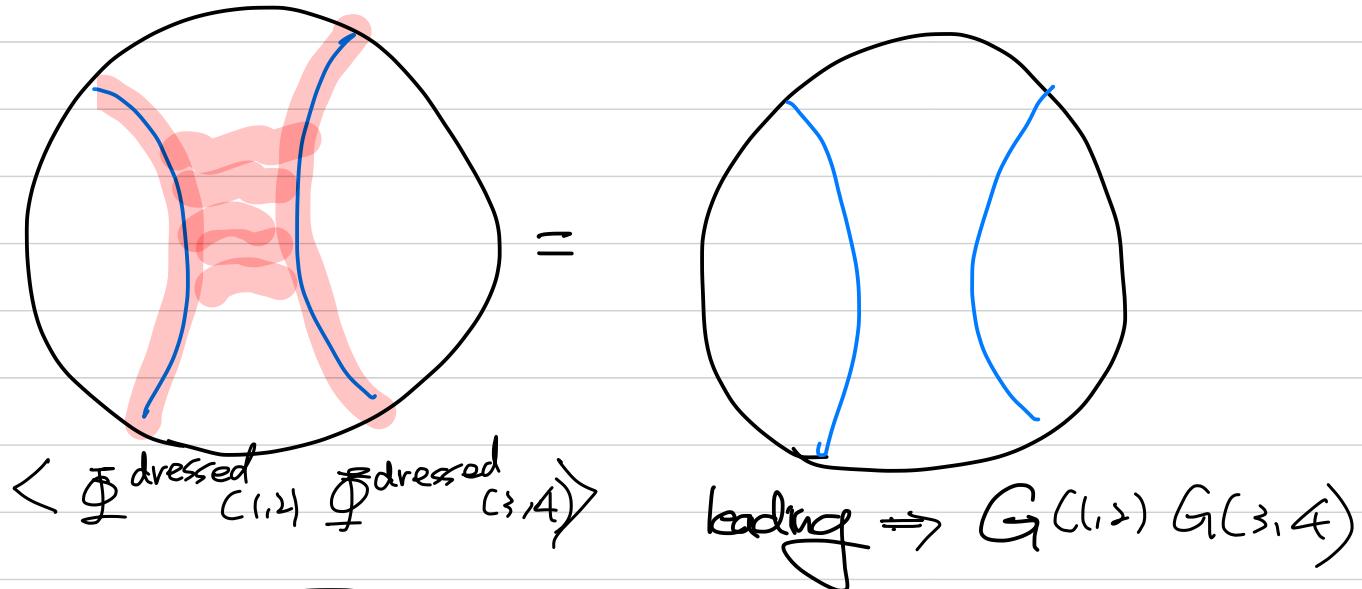
$$g_{s,n}(\sigma) = 0 \quad \text{for } n = 0, \pm 1, \pm 2, \dots, \pm s-1.$$

$$\langle f_{s,n}, f_{s',n'} \rangle \sim f_{ss'} f_{nn'}$$

\Rightarrow Solution for recursion relations.

$$g_{s,n}(\sigma) = \begin{cases} \frac{[(s-1)!]^2}{(2s-3)!!} \frac{e^{i\pi s}}{\sin \sigma} \left(e^{in\sigma} P_{s-1}^{n,-n}(\sigma) - e^{-in\sigma} P_{s-1}^{-n,n}(\sigma) \right) & \text{for } |n| \geq s \\ 0 & \text{for } |n| < s \end{cases}$$

* Four point functions.



one spin-s field exchange.

$$\langle S_n^{(s)} S_{-n}^{(s)} \rangle \underset{\frac{1}{C} \sum \frac{1}{n^2(n^2-1)\dots(n^2-(s-1)^2)}}{\underset{\text{analytic continuation}}{\sim}} S_n^{(s)} \Phi_{\text{dressed}}^{(s)} S_{-n}^{(s)} \Phi_{\text{dressed}}^{(s)}.$$

$$\text{OTOC} \sim e^{\frac{2\pi i}{\beta}(s-1)t}.$$

$$\text{Lyapunov exponent : } \frac{2\pi}{\beta} (s-1)$$

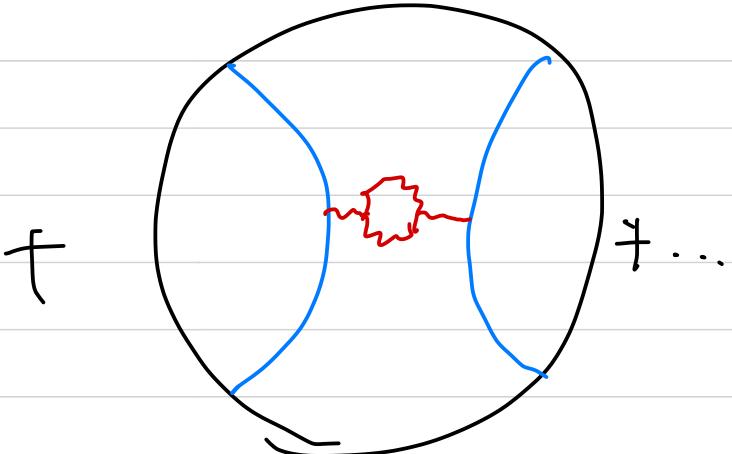
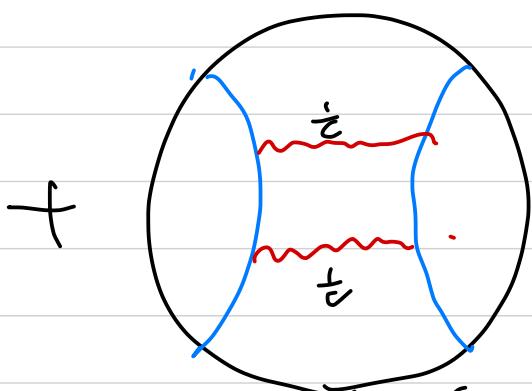
: violate the bound on chaos (for $s > 2$)

\Rightarrow Semi-classical limit : non-unitary.

Interesting to study $\frac{1}{c}$ corrections. ($\frac{1}{c}$ correction to Lyapunov exp)

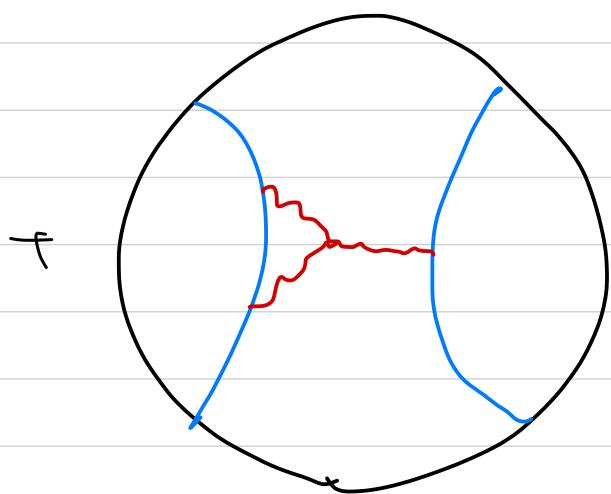
$$\langle \xi_n \xi_{-n} \rangle \underset{(S=2)}{\sim} \frac{1}{c^2} + \frac{1}{c^2} (\dots) + \dots$$

loop correction.

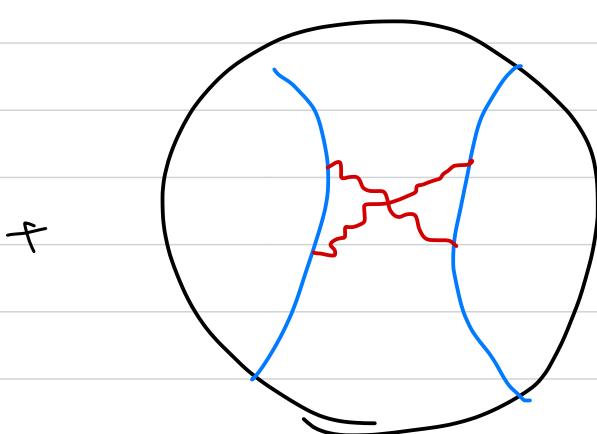


$$\langle \xi_n \xi_{-n} \rangle \underset{\frac{1}{c}}{\sim} \frac{1}{c} + \dots$$

$$\frac{1}{c^4} \cdot c^2 \sim \frac{1}{c^2}$$



$$\frac{1}{c^3} \cdot c \sim \frac{1}{c^2}$$



$$\frac{1}{c^4} \cdot c \sim \frac{1}{c^3}$$

$$\langle \xi \xi \xi \rangle \xi \xi \bar{\xi} \bar{\xi} .$$