Formal Higher Spin Gravity

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Based mostly on 1901.01426 with Alexei Sharapov

Zhenya Skvortsov, Albert Einstein Institute

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Main Messages

- Higher spin gravities (HSGRA), hypothetical theories with graviton and massless higher spin fields, have been studied for many years, but until recently there has not been a single example worked out in detail (action, quantization, ...)

- Thinking of tensionless strings or of AdS/CFT allows one to see that HSGRA are not quite conventional field theories and have to have severe non-localities which are not well-understood at present (invalidate the Noether procedure)

- There is a way to get a nontrivial result by abandoning locality — formal HSGRA following (Vasiliev, 1988). We find the general solution to this problem by explicitly constructing all interaction vertices and clarifying their origin and relation to Deformation Quantization

Main Messages

Massless higher spin fields are not easy to make interact (Weinberg, Coleman-Mandula, ..., Maldacena-Zhiboedov, Taronna-Sleight). The main obstacle is the lack of the effective field theory approximation with finitely many fields and derivatives.

At present there are only three **well-defined** examples

- generalization of the $3d$ Chern-Simons formulation of gravity (Blencowe; Campoleoni, Fredenhagen, Pfenninger, Theisen; Henneaux, Rey);
- Conformal Higher Spin Gravity — extension of the conformal gravity (Tseytlin, Segal; Bekaert, Joung, Mourad; Joung, Nakach, Tseytlin; Adamo, Tseytlin), quantization seems to work
- Chiral Higher Spin Gravity (Flat and AdS) (Metsaev; Ponomarev, E.S.; Ponomarev; E.S., Tran, Tsulaia), quantization and AdS_4/CFT_3

Let's abandon action, quantization and talk about equations of motion...

Formal Equations

Let's relax the assumptions even further and talk about formal equations:

$$d\Phi = \mathcal{V}_2(\Phi,\Phi) + \mathcal{V}_3(\Phi,\Phi,\Phi) + ... = F(\Phi)$$

Here

- d is some formal differential $dd \equiv 0$;
- sometimes we can make it into $d = dx^m\partial_m$, but in general this is dangerous as the equations may not be well-defined as PDE's/field theory, hence **formal**
- F is constrained by $dd \equiv 0$. Defining $Q = F\frac{\delta}{\delta\Phi}$, the integrability implies $QQ = 0$ and we have Q-manifold or better say L_∞-algebra (Free Differential Algebras is an obsolete term)

Formal Equations

Therefore, we are looking for

$$d\Phi = \mathcal{V}_2(\Phi, \Phi) + \mathcal{V}_3(\Phi, \Phi, \Phi) + ... = F(\Phi)$$

where Φ is some set of fields, $\Phi \equiv \Phi^A e_A$, where e_A is a convenient base

All the info is in the structure functions

$$F \equiv F^A e_A \qquad\qquad F^A(\Phi) = \sum F^A_{B_1 ... B_n} \Phi^{B_1} ... \Phi^{B_n}$$

where the Taylor coefficients $F^A_{B_1 ... B_n}$ are just constants to be found

The only constraint is the formal consistency:

$$0 \equiv dd\Phi = dF \qquad\qquad \Longleftrightarrow \qquad\qquad F^B \frac{\delta}{\delta\Phi^B} F^A \equiv 0$$

Therefore, we are looking for

$$d\Phi = \mathcal{V}_2(\Phi, \Phi) + \mathcal{V}_3(\Phi, \Phi, \Phi) + ... = F(\Phi)$$

where the consistency conditions

$$F^B \frac{\delta}{\delta \Phi^B} F^A \equiv 0$$

are completely disentangled from the space-time dependence (it does not matter if d is the exterior derivative on some manifold and it also does not matter what the dimension of the manifold is)

Therefore, we are looking for

$$d\Phi = \mathcal{V}_2(\Phi, \Phi) + \mathcal{V}_3(\Phi, \Phi, \Phi) + ... = F(\Phi)$$

such that the system is formally consistent

$$F^B \frac{\delta}{\delta \Phi^B} F^A \equiv 0$$

The simplest example of such a system is flat connection

$$d\omega^i = f^i_{jk}\, \omega^j \wedge \omega^k \qquad \Longleftrightarrow \qquad f^i_{jk} = \text{Lie algebra}$$

where $\Phi = \{\omega^i \equiv \omega^i_\mu\, dx^\mu\}$ is a one-form connection.

The formal consistency is equivalent to Jacobi identities for f^i_{jk}

Formal Equations

Therefore, we are looking for

$$d\Phi = \mathcal{V}_2(\Phi, \Phi) + \mathcal{V}_3(\Phi, \Phi, \Phi) + ... = F(\Phi)$$

The initial data, which will give us $\mathcal{V}_2(\Phi, \Phi)$ can be found on the CFT side

Higher Spin Algebras

Since masslessness/gauge invariance of Higher Spin Gravities is AdS/CFT dual to conservation, the dual CFT's should have conserved higher rank tensors. This, in $d > 2$, is possible only for free CFT's (or to the leading order in $1/N$ for some interacting CFT's like vector models), (Maldacena, Zhiboedov; Boulanger, Ponomarev, E.S., Taronna; Alba, Diab)

There are multiple ways to see that any free CFT gives us an infinite-dimensional, **associative algebra** that contains $so(d,2)$ as Lie subalgebra, which we call Higher Spin Algebra (HSA)

The simplest way is to think of $U(so(d,2))/I$, i.e. functions of Lorentz generators L_{ab} and translations P_a with some additional relations

There are many different HSA's known and many of them can be realized in terms of Weyl algebras, i.e. functions $f(p,q)$ where $[p_i, q^j] = \delta^i_j$

Formal HSGRA problem

From now on we can forget about the HS motivation and assume that we are given any **associative** algebra A

$$a \star (b \star c) = (a \star b) \star c \qquad\qquad a, b, c \in A$$

It turns out that the whole problem of formal HSGRA does not go beyond this concept!

We would like to define $\Longrightarrow$ below

free CFT		formal HSGRA
$\Updownarrow$	$\Longrightarrow$	$\Updownarrow$
HS algebra, A		$d\Phi = \mathcal{V}_2(\Phi, \Phi) + ...$

by constructing all vertices $\mathcal{V}_n$ from any given HSA

HS symmetry is a global symmetry on the CFT side, so it should be gauged in AdS. A natural way to start gauging a symmetry is to take a connection $\omega \equiv \omega_\mu(x)dx^\mu$ valued in a given algebra.

The simplest equation possible

$$d\omega = \frac{1}{2}[\omega, \omega]_\star = \omega \star \omega$$

already describes maximally symmetric HS backgrounds, in particular AdS_{d+1} since $so(d, 2)$ is a subalgebra. Flatness of

$$\omega = e^a P_a + \frac{1}{2}\varpi^{a,b} L_{ab}$$

gives vielbein $e^a \equiv e^a_\mu \, dx^\mu$ and spin-connection $\varpi^{a,b}$ of AdS_{d+1}. In $3d$ HS Black Holes are described by the same equation

Toy-model

HS symmetry is a global symmetry on the CFT side, so it should be gauged in AdS. A natural way to start gauging a symmetry is to take a connection $\omega \equiv \omega_\mu(x)dx^\mu$ valued in a given algebra.

The simplest equation possible

$$d\omega = \frac{1}{2}[\omega, \omega]_\star = \omega \star \omega$$

is sensitive only to the Lie part of the associative HS algebra A. Commutator $[a, b]_\star$ forgets the symmetric part of the structure constants. If we take ω to be matrix-valued (matrix of HS algebra elements) then we can gauge YM groups and the same time won't forget the underlying associative structure

$$\text{Lie}(A \otimes \mathrm{Mat}_M) \qquad \Longleftrightarrow \qquad A$$

after applying the math spell 'Morita invariance'

Let's try to deform the simplest equation possible (toy-model)

$$d\omega = \omega \star \omega + \phi(\omega, \omega)\hbar + ...$$

The undeformed equation was (after the matrix extension trick) equivalent to saying that $\star$ defines an associative product. Not surprisingly the 'interacting equation' is equivalent to associativity of the deformed product

$$a \circ b = a \star b + \phi(a, b)\hbar + ... \qquad a \circ (b \circ c) = (a \circ b) \circ c$$

Mathematically, ϕ is a nontrivial Hochschild two-cocycle

$$-a \star \phi(b, c) + \phi(a \star b, c) - \phi(a, b \star c) + \phi(a, b) \star c = 0$$

Therefore, our 'interacting theory' is equivalent to deforming the product in a given HS algebra A. This toy-model is not so toyish ...

Formal HSGRA Problem, Vasiliev 1988

For HSGRA the string field Φ consists of

- a connection ω (meant to gauge a given HS-algebra A);
- an additional matter-like field, zero-form C also taking values in A

Thinking of a free CFT with fundamental field φ furnishing some irreducible representation S of $so(d,2)$, Flato-Fronsdal-type theorems say that

- $S \otimes S^*$ (ket-bra-type matrices) is, roughly speaking, how HS-algebra looks like and this is where ω takes values
- $S \otimes S$ gives the spectrum of all bi-linear operators (e.g. HS-currents $J_s = \varphi \partial^s \varphi$). These are the same irreps as massless HS fields in AdS and this is where C should take values

Formal HSGRA Problem, Vasiliev 1988

For HSGRA the string field Φ consists of

- a connection ω (meant to gauge a given HS-algebra A);
- an additional matter-like field, zero-form C also taking values in A

The initial vertex $\mathcal{V}_2$ is

$$d\omega = \omega \star \omega$$
$$dC = \omega \star C - C \star \pi(\omega)$$

Again, very roughly $S \otimes S^*$ is one-to-one with $S \otimes S$, while the $S^* \to S$ is realized by certain automorphism π:

$$\pi(L_{ab}) = L_{ab} \qquad \pi(P_a) = -P_a$$

that extends to any given HS-algebra

Formal HSGRA Problem, Vasiliev 1988

To summarize, $\Phi = \{\omega, C\}$ both taking values in any given HS-algebra A. The 'free' equations are

$$d\omega = \omega \star \omega$$
$$dC = \omega \star C - C \star \pi(\omega)$$

The first equation describes maximally-symmetric HS-backgrounds, in particular, AdS.

The second equation (over AdS) describes HS fields

$$C = C(P, L) = \Phi 1 + F^{ab} L_{ab} + C^{ab,cd} L_{ab} L_{cd} + ...$$

where Φ is a scalar field, F^{ab} is the Maxwell tensor, $C^{ab,cd}$ is the Weyl tensor and ... contains HS generalizations and auxiliary fields

Formal HSGRA Problem, Vasiliev 1988

To summarize, $\Phi = \{\omega, C\}$ both taking values in any given HS-algebra A (any associative algebra). The 'free' equations are

$$d\omega = \omega \star \omega$$
$$dC = \omega \star C - C \star \pi(\omega)$$

We would like to construct interaction vertices

$$d\omega = \omega \star \omega + \mathcal{V}_3(\omega, \omega, C) + ...$$
$$dC = \omega \star C - C \star \pi(\omega) + \mathcal{V}_3(\omega, C, C) + ...$$

General Solution of the HSGRA problem

We would like to construct interaction vertices

$$
\begin{aligned}
d\omega &= \omega \star \omega + \mathcal{V}_3(\omega,\omega,C) + ... \\
dC &= \omega \star C - C \star \pi(\omega) + \mathcal{V}_3(\omega,C,C) + ...
\end{aligned}
$$

Step 1 is to make the problem even more general by hiding π inside a slightly bigger algebra — crossed product $A \ltimes \mathbb{Z}_2$ where $\mathbb{Z}_2 = \{1,\pi\}$, $\pi^2 = 1$. Its elements are pairs $a = a'1 + a''\pi$. The twisted commutator is now a part of the usual commutator in $A \ltimes \mathbb{Z}_2$ and we can erase it

$$
\begin{aligned}
d\omega &= \omega \star \omega + \mathcal{V}_3(\omega,\omega,C) + ... \\
dC &= \omega \star C - C \star \omega + \mathcal{V}_3(\omega,C,C) + ...
\end{aligned}
$$

General Solution of the HSGRA problem

We would like to construct interaction vertices

$$d\omega = \omega \star \omega + \mathcal{V}_3(\omega, \omega, C) + ...$$
$$dC = \omega \star C - C \star \omega + \mathcal{V}_3(\omega, C, C) + ...$$

where the first vertices are determined by an associative algebra A

$$a \star (b \star c) = (a \star b) \star c \qquad\qquad a, b, c \in A$$

In the HS case it should be $B \ltimes \mathbb{Z}_2$ for some HS-algebra B), but the problem is more general right now.

The main result is that

formal HSGRA		deformed product in A
$\Updownarrow$	$\Longleftrightarrow$	$\Updownarrow$
$d\Phi = \mathcal{V}_2(\Phi,\Phi) + ...$		$a \circ b = a \star b + \phi_1(a,b)\hbar + ...$

Indeed, assume that the algebra A can be deformed as an associative one

$$a \circ (b \circ c) = (a \circ b) \circ c \qquad\qquad a \circ b = a \star b + \sum_k \phi_k(a,b)\hbar^k$$

Then, we can write all vertices explicitly as

$$\mathcal{V}_3(\omega,\omega,C) = \phi_1(\omega,\omega) \star C$$
$$\mathcal{V}_4(\omega,\omega,C,C) = \phi_2(\omega,\omega) \star C \star C + \phi_1(\phi_1(\omega,\omega),C) \star C$$

and so on

General Solution of the HSGRA problem

The main result is that

formal HSGRA		deformed product in A
$\Updownarrow$	$\Longleftrightarrow$	$\Updownarrow$
$d\Phi = \mathcal{V}_2(\Phi, \Phi) + ...$		$a \circ b = a \star b + \phi_1(a, b)\hbar + ...$

Therefore, the wild problem of constructing infinitely many of multi-linear maps, the vertices $\mathcal{V}_n(...)$, turns out to be equivalent to a much more simple problem of deforming the product in a given associative algebra A

We can write $\mathcal{V}_n$ as sums over certain trees or as a simple equation that generates them order by order from $\circ$-product

Essentially, Formal HSGRA is a way to repackage the Taylor coefficients ϕ_k of the deformed product and write the associativity condition as equations of motion!

formal HSGRA		deformed product in A
$\Updownarrow$	$\Longleftrightarrow$	$\Updownarrow$
$d\Phi = \mathcal{V}_2(\Phi, \Phi) + \dots$		$a \circ b = a \star b + \phi_1(a, b)\hbar + \dots$

Since Formal HSGRA = associativity of $\circ$, other problems can be reduced to purely algebraic ones of the deformed algebra $A_\hbar$, e.g.

$$\text{Observables} \qquad \Longleftrightarrow \qquad \text{Invariants of } A_\hbar$$

It is quite easy to construct certain observables, including the new ones. For AdS/CFT correlation functions, the most natural conjecture is

$$\mathrm{tr}(C \circ \dots \circ C)$$

at $\hbar = 0$ we definitely have the free CFT correlation functions (Colombo, Sundell; Didenko, E.S.; De Filippi, Bonezzi, Boulanger, Sundell)

Formally Bad News

In $3d$ the right starting point is scalar+HS:

$$dA_{\pm} = A_{\pm} \star A_{\pm}$$
$$dC = A_{+} \star C - C \star A_{-}$$

- there are no AAC corrections since scalar does not couple to Fronsdal fields linearly
- there are no AACC corrections since all stress-tensors are formally exact (Prokushkin, Vasiliev; Lucena-Gomez, Kessel, E.S., Taronna)

The only formal deformation is to shift λ to $\lambda + \delta\lambda$, which leads to the same system (more or less in Prokushkin, Vasiliev).

The system is also inconsistent with AdS/CFT — since there is no backreaction from the scalar field to the A-sector

List of Formal HSGRA

- $4d$ HSGRA (Vasiliev, 90, 91)
- $3d$ (Prokushkin, Vasiliev, 98)
- Type-A in any d (Vasiliev, 03)
- partially-massless Type-A (Alkalaev, Grigoriev, E.S., 14)
- another realization of the Type-A (Bekaert, Grigoriev, E.S., 2017)
- Type-B in any d (Grigoriev, E.S., 18)
- another form of Type-B (Sharapov, E.S., 19)
- toy-models in $5d$ (Sharapov, E.S., 19)
- ....

HSGRA discussion naturally leads to a vast extension of the usual deformation quantization:

$$f \star g = f \cdot g + \hbar\{f, g\} + \text{Kontsevich}$$

In fact, the true statement is that the DGLA of polydifferential operators is formal, i.e. quasi-isomorphic to its cohomology, which is the DGLA of polyvector fields. In particular, there is a unique quantization of the algebra of functions $C(M)$

Deformation Quantization

HSGRA discussion naturally leads to a vast extension of the usual deformation quantization:

$$f \star g = f \cdot g + \hbar\{f, g\} + \text{Kontsevich}$$

HS-algebras are all obtained as DQ of functions on the coadjoint orbit that corresponds to the irrep of $so(d, 2)$ that we call free field

However, usually HS algebras are rigid and cannot be deformed

The deformation is along the π-map!

The statement is that $A \ltimes \mathbb{Z}_2$ is not rigid while A is rigid whenever A is a HS algebra

Note that $A \ltimes \mathbb{Z}_2$ is slightly non-commutative even if A is commutative we are lead to consider Poisson manifolds equipped with some discrete group of symmetries Γ and instead of $C(M)$ we need crossed product $C(M) \ltimes \Gamma$ or orbifold $C(M)/\Gamma$.

It seems that the formality conjecture extends to this case as well. The new feature is that there are completely new, independent directions of quantization, see e.g. Sharapov, E.S. for Weyl $\ltimes$ Symplectic reflections

For HSGRA we have a two-parameter family of algebras: h_K that constructs HS-algebra from the classical phase space (we usually ignore h_K) and another $\hbar$ which is due to quantizing along π

We show that everything is encoded in

$$a \circ b \qquad\qquad \circ = \circ \hbar_K, \hbar_{HS}$$

Concluding Remarks

- **Formal HSGRA** $\equiv$ **deformed HS algebra**. Construct your own HSGRA(?) by taking any family of algebras

- formal equations can be also be solved and are equivalent to the well-defined Lax equations (in the paper)

- well-defined observables, some of which are new. Holographic correlation functions $\rightarrow 3d$ bosonization

- Formal HSGRA points towards a more general deformation quantization setup — Poisson orbifolds can have new directions of quantization — extension of the Kontsevich formality. Topological open strings are behind the corner ...

Thank you for your attention!