$\mathcal{W}$-symmetry and instanton R-matrix

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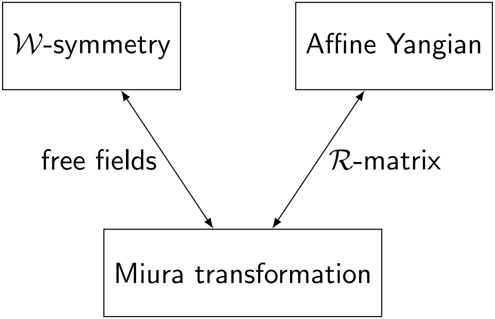
March 21, 2019

Plan

'algebraic structures in 2d conformal field theory with higher spins'



$\mathcal{W}_\infty$: (spherical) degenerate double affine Hecke algebra (Cherednik), cohomological Hall algebra (Kontsevich & Soibelman), Yangian of $\widehat{\mathfrak{gl}(1)}$, quantum toroidal algebra, Ding-Iohara-Miki algebra

W algebras

$\mathcal{W}$-algebas: extensions of the Virasoro algebra (2d CFT) by higher spin currents - appear in many different contexts:

- integrable hierarchies of PDE (KdV, KP)
- (old) matrix models
- instanton partition functions and AGT
- holographic dual description of 3d higher spin theories
- quantum Hall effect
- topological strings
- higher spin square (Gaberdiel, Gopakumar)
- $\mathcal{N} = 4$ SYM at junction of three codimension 1 defects (Gaiotto, Rapčák)
- geometric representation theory (equivariant cohomology of moduli spaces)

Zamolodchikov $\mathcal{W}_3$ algebra

$\mathcal{W}_3$ algebra constructed by Zamolodchikov (1984) has a stress-energy tensor (Virasoro algebra) with OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg.$$

together with spin 3 primary field $W(w)$

$$T(z)W(w) \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + reg.$$

To close the algebra we need to find the OPE of W consistent with associativity.

The result:

$$\begin{aligned} W(z)W(w) \quad \sim \quad & \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\ & + \frac{1}{(z-w)^2}\left(\frac{32}{5c+22}\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right) \\ & + \frac{1}{z-w}\left(\frac{16}{5c+22}\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w)\right) \end{aligned}$$

Λ is a quasiprimary 'composite' (spin 4) field,

$$\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z).$$

The algebra is non-linear, not a Lie algebra in the usual sense (linearity should not be expected for spins ≥ 3).

$\mathcal{W}_\infty$

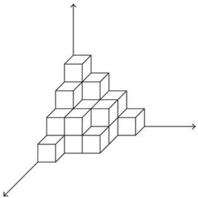
- $\mathcal{W}_\infty$: generalizing $\mathcal{W}_3$ to fields of spin $2, 3, 4, 5, \ldots$
- solving associativity conditions for this field content $\rightsquigarrow$ two-parameter family (Gaberdiel-Gopakumar)
- parameters: central charge c and rank parameter λ
- choosing $\lambda = N \to$ truncation of $\mathcal{W}_\infty$ to $\mathcal{W}_N = \mathcal{W}[\mathfrak{sl}(N)]$, i.e. $\mathcal{W}_\infty$ is interpolating algebra for the whole A_{N-1} series of $\mathcal{W}_N$ algebras (cf. $\mathfrak{sl}(N)$ vs $hs[\lambda]$ in $3d$ higher spin gravities)
- adding spin 1 field, we have $\mathcal{W}_{1+\infty} \rightsquigarrow$ many simplifications
- surprise: $\mathcal{W}_{1+\infty}$ contains all $\mathcal{W}[\mathfrak{g}]$ for all simple $\mathfrak{g}$ (except possibly $\mathfrak{f}_4$?)

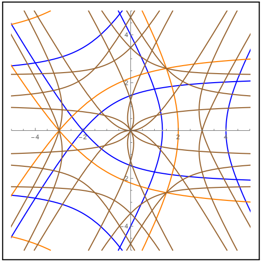
- triality symmetry of the algebra (Gaberdiel & Gopakumar)

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0 \quad c = (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1)$$

- MacMahon function as vacuum character of the algebra

$$\prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + \cdots$$





The resulting OPEs look like

$$
\begin{aligned}
U_3(z)U_5(w) \quad \sim \quad & \frac{1}{(z-w)^7}\Big(\frac{1}{2}\alpha(n-3)(n-2)(n-1)n\left(4\alpha^2\left(\alpha^2(n(5n-9)+1)-3n+4\right)+1\right)\mathbb{1}\Big) \\
& +\frac{1}{(z-w)^6}\Big(\frac{1}{6}(n-3)(n-2)(n-1)\left(6\alpha^4 n(2n-3)+\alpha^2(10-9n)+1\right)U_1(w)\Big) \\
& +\frac{1}{(z-w)^5}\Big(-\alpha(n-3)(n-2)(n-1)\left(-4\alpha^2+3\alpha^2 n-1\right)(U_1U_1)(w) \\
& +\alpha(n-3)(n-2)\left(4\alpha^2 n^2-4\alpha^2 n-n-2\right)U_2(w) \\
& -\frac{1}{2}\alpha^2(n-3)(n-2)(n-1)\left(4\alpha^2 n(2n-3)-3n+2\right)U_1'(w)\Big) \\
& +\frac{1}{(z-w)^4}\Big(-\alpha(n-3)(n-2)(n-1)\left(\alpha^2(3n-4)-1\right)(U_1'U_1)(w) \\
& -\frac{1}{2}(n-3)(n-2)\left(2\alpha^2(n-1)-1\right)(U_1U_2)(w) \\
& +(n-3)\left(\alpha^2\left(n^2+2\right)-3\right)U_3(w) \\
& -\frac{1}{4}\alpha^2(n-3)(n-2)(n-1)\left(4\alpha^2 n(2n-3)-3n+2\right)U_1''(w) \\
& +\alpha(n-3)(n-2)\left(\alpha^2(n-1)n-1\right)U_2'(w)\Big) \\
& +\cdots
\end{aligned}
$$

But there are special properties of OPE that allow guessing a general formula

- a basis of local fields $U_j(z)$ in which the OPEs have purely quadratic non-linearity (impossible in primary basis)
- all the derivative terms can be resummed and we find a bilocal expansion (Lukyanov)

$$U_j(z)U_k(w) + \sum_{l+m<j+k} C_{jk}^{lm}(N,\alpha_0)\frac{U_l(z)U_m(w)}{(z-w)^{j+k-l-m}} \sim reg.$$

- other special properties like $C_{jk}^{lm}(N) = C_{j+1k+1}^{l+1m+1}(N+1)$
- how to explain these?

Yangian of $\widehat{\mathfrak{gl}(1)}$

The Yangian of $\widehat{\mathfrak{gl}(1)}$ (Arbesfeld-Schiffmann-Tsymbaliuk) is an associative algebra with generators $\psi_j, e_j, f_j, j \geq 0$ and relations

$$
\begin{aligned}
0 &= [e_{j+3}, e_k] - 3\,[e_{j+2}, e_{k+1}] + 3\,[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}] \\
&\quad +\sigma_2\,[e_{j+1}, e_k] - \sigma_2\,[e_j, e_{k+1}] - \sigma_3\,\{e_j, e_k\} \\
0 &= [f_{j+3}, f_k] - 3\,[f_{j+2}, f_{k+1}] + 3\,[f_{j+1}, f_{k+2}] - [f_j, f_{k+3}] \\
&\quad +\sigma_2\,[f_{j+1}, f_k] - \sigma_2\,[f_j, f_{k+1}] + \sigma_3\,\{f_j, f_k\} \\
0 &= [\psi_{j+3}, e_k] - 3\,[\psi_{j+2}, e_{k+1}] + 3\,[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}] \\
&\quad +\sigma_2\,[\psi_{j+1}, e_k] - \sigma_2\,[\psi_j, e_{k+1}] - \sigma_3\,\{\psi_j, e_k\} \\
0 &= [\psi_{j+3}, f_k] - 3\,[\psi_{j+2}, f_{k+1}] + 3\,[\psi_{j+1}, f_{k+2}] - [\psi_j, f_{k+3}] \\
&\quad +\sigma_2\,[\psi_{j+1}, f_k] - \sigma_2\,[\psi_j, f_{k+1}] + \sigma_3\,\{\psi_j, f_k\} \\
0 &= [\psi_j, \psi_k] \\
\psi_{j+k} &= [e_j, f_k]
\end{aligned}
$$

'initial/boundary conditions'

$$[\psi_0, e_j] = 0, \qquad [\psi_1, e_j] = 0, \qquad [\psi_2, e_j] = 2e_j,$$
$$[\psi_0, f_j] = 0, \qquad [\psi_1, f_j] = 0, \qquad [\psi_2, f_j] = -2f_j$$

and finally the Serre relations

$$0 = \mathrm{Sym}_{(j_1,j_2,j_3)} [e_{j_1}, [e_{j_2}, e_{j_3+1}]], \qquad 0 = \mathrm{Sym}_{(j_1,j_2,j_3)} [f_{j_1}, [f_{j_2}, f_{j_3+1}]].$$

There are 3 parameters of the algebra, $h_1, h_2, h_3 \in \mathbb{C}$ constrained by $h_1 + h_2 + h_3 = 0$. Furthermore, we used the shorthand notation

$$\begin{aligned} \sigma_2 &= h_1 h_2 + h_1 h_3 + h_2 h_3 \\ \sigma_3 &= h_1 h_2 h_3. \end{aligned}$$

We have both commutators and anticommutators in defining quadratic relations (but no $\mathbb{Z}_2$ grading).

Introducing generating functions

$$e(u) = \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}}, \qquad f(u) = \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}}, \qquad \psi(u) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}$$

simplifies the first set of formulas above (almost!!) to

$$\begin{aligned} e(u)e(v) &= \varphi(u-v)e(v)e(u), \qquad & f(u)f(v) &= \varphi(v-u)f(v)f(u), \\ \psi(u)e(v) &= \varphi(u-v)e(v)\psi(u), \qquad & \psi(u)f(v) &= \varphi(v-u)f(v)\psi(u) \end{aligned}$$

with structure function

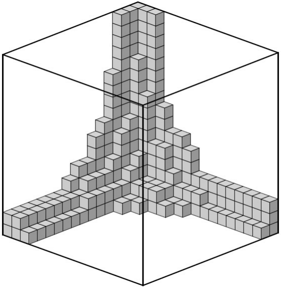
$$\varphi(u) = \frac{(u+h_1)(u+h_2)(u+h_3)}{(u-h_1)(u-h_2)(u-h_3)}$$

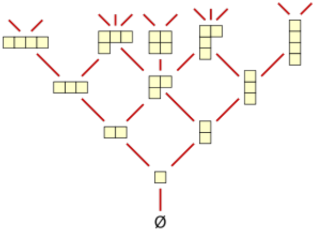
The representation theory of the algebra is much simpler in this Yangian formulation and it is controlled by this simple function.

$\psi(u), e(u)$ and $f(u)$ in representations act like

$$\psi(u)\,|\Lambda\rangle = \psi_0(u) \prod_{\square\in\Lambda} \varphi(u-h_\square)\,|\Lambda\rangle\,, \quad e(u)\,|\Lambda\rangle = \sum_{\square\in\Lambda^+} \frac{E(\Lambda\to\Lambda+\square)}{u-h_\square}\,|\Lambda+\square\rangle$$

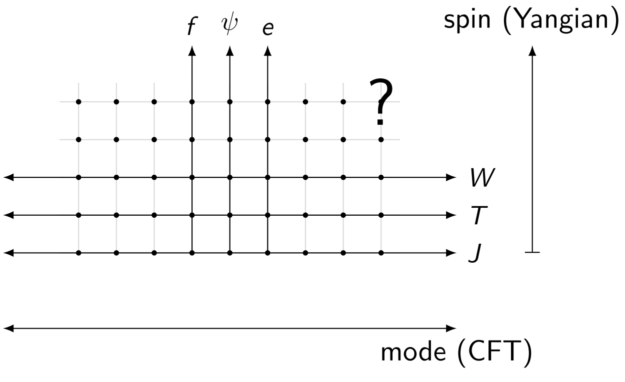
where the states $|\Lambda\rangle$ are associated to geometric configurations of boxes (plane partitions,...) and where $h_\square = \sum_j h_j x_j(\square)$ is the weighted geometric position of the box.





Two different descriptions of the algebra:

- usual CFT point of view with local fields $J(z), T(z), W(z), \ldots$ with increasingly complicated OPE as we go to higher spins
- Yangian point of view (Arbesfeld-Schiffmann-Tsymbaliuk) where all the spins are included in the generating functions $\psi(u), e(u)$ and $f(u)$ but accessing higher mode numbers is difficult



How to connect these two descriptions?

- the generators with low spin and mode number can be identified, i.e. $\psi_2 = 2L_0, e_0 = J_{-1}, f_0 = -J_{+1}$
- $\psi_3 = 3W_0 + \ldots + \sigma_3 \sum_{m>0}(3m-1)J_{-m}J_m$ (cut & join operator, not a zero mode of a local field!)
- this is sufficient to find the map in principle, but what is the more conceptual way to understand the map?

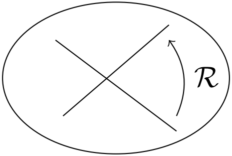
Miura transformation and $\mathcal{R}$-matrix

- Miura transformation - free field representation of $\mathcal{W}_N$

$$(\partial + \partial\phi_1(z)) \cdots (\partial + \partial\phi_N(z)) = \sum_{j=0}^{N} U_j(z)\partial^{N-j}$$

- quantization of differential operators
- the embedding of $\mathcal{W}_N$ in the bosonic Fock space depends on the way we order the fields
- Maulik-Okounkov: $\mathcal{R}$-matrix as transformation between two equivalent orderings

$$(\partial + \partial\phi_1)(\partial + \partial\phi_2) = \mathcal{R}^{-1}(\partial + \partial\phi_2)(\partial + \partial\phi_1)\mathcal{R}$$

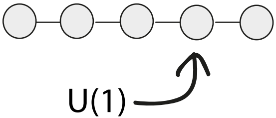


- $\mathcal{R}$ defined in this way satisfies the Yang-Baxter equation (two ways of reordering $321 \to 123$)

$$\mathcal{R}_{12}(u_1 - u_2)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{23}(u_2 - u_3) = \\ = \mathcal{R}_{23}(u_2 - u_3)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{12}(u_1 - u_2)$$

- the spectral parameter u - the global $\mathfrak{u}(1)$ charge
- $\mathcal{R}$-matrix satisfying YBE $\rightsquigarrow$ apply the algebraic Bethe ansatz

W5:



- spin chain of length $N \rightsquigarrow \mathcal{W}_N$ algebra

- consider an *auxiliary* Fock space $\mathcal{F}_A$ and a *quantum* space $\mathcal{F}_Q \equiv \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_N$
- we associate to this the monodromy matrix $\mathcal{T}_{AQ} : \mathcal{F}_A \otimes \mathcal{F}_Q \to \mathcal{F}_A \otimes \mathcal{F}_Q$ defined as

$$\mathcal{T}_{AQ} = \mathcal{R}_{A1} \mathcal{R}_{A2} \cdots \mathcal{R}_{AN}$$

- if the individual $\mathcal{R}$-matrices satisfied the YBE, $\mathcal{T}$ will also satisfy YBE with respect to two auxiliary spaces A and B

$$\mathcal{R}_{AB} \mathcal{T}_A \mathcal{T}_B = \mathcal{T}_B \mathcal{T}_A \mathcal{R}_{AB}$$

- the algebra of matrix elements of $\mathcal{T}$ satisfying this equation is the Yangian (quadratic defining relations!)

- in our situation the $\mathcal{R}$-matrix controlling the commutation relations of Yangian has rows and columns labeled by Young diagrams
- we can however restrict to simple matrix elements, i.e.

$$\mathcal{H} = \langle 0|_A \mathcal{T} |0\rangle_A , \quad \mathcal{E} = \langle 0|_A \mathcal{T} |1\rangle_A , \quad \mathcal{F} = \langle 1|_A \mathcal{T} |0\rangle_A$$

- the YBE now implies relations between these operators like

$$0 = [\mathcal{H}(u), \mathcal{H}(v)]$$

(infinite set of commuting Hamiltonians) or

$$(u - v + h_3)\mathcal{H}(u)\mathcal{E}(v) = (u - v)\mathcal{E}(v)\mathcal{H}(u) + h_3\mathcal{H}(v)\mathcal{E}(u)$$

(ladder operators)

- How are these generating functions related to AST?

$$\psi(u) = \frac{u + \sigma_3 \psi_0}{u} \frac{\mathcal{H}(u + h_1)\mathcal{H}(u + h_2)}{\mathcal{H}(u)\mathcal{H}(u + h_1 + h_2)}$$

$$e(u) = h_3^{-1} \mathcal{H}(u)^{-1} \mathcal{E}(u), \qquad f(u) = -h_3^{-1} \mathcal{F}(u) \mathcal{H}(u)^{-1}$$

- Verification of the relations satisfied by these is difficult and MO Yangian seems in fact to be bigger than AST Yangian (Schiffmann: differ by central elements)

How does the $\mathcal{R}$-matrix look like?

- a fermionic form obtained by A. Smirnov by studying the large N limit of $\mathfrak{gl}(N)$ $\mathcal{R}$-matrix - complicated
- a bosonic formula is not known so far, but results of Nazarov-Sklyanin can be interpreted as matrix elements of $\mathcal{R}$-matrix in mixed representation where the two bosonic Fock spaces are associated to different asymptotic directions of $\mathcal{W}_\infty$

$$
\begin{aligned}
\mathcal{R}(u) \quad = \quad & \mathbb{1} - \frac{1}{u} \sum_{j_1>0} a^-_{-j_1} a^-_{j_1} \\
& + \frac{1}{2! u(u+1)} \sum_{j_1,j_2>0} (a^-_{-j_1} a^-_{-j_2} + a^-_{-j_1-j_2})(a^-_{j_1} a_{j_2} + a^-_{j_1+j_2}) \\
& + \ldots
\end{aligned}
$$

with $a^-_j \equiv a^{(1)}_j - a^{(2)}_j$.

Application: Calogero models

- instead of building the monodromy matrix $\mathcal{T}$ from the Fock-Fock $\mathcal{R}$-matrices, we could use the Miura operators $\partial + \partial\phi(z)$, using for the quantum space the space of functions of z coordinate
- this produces for free commuting Hamiltonians of quantum Calogero models as well as ladder operators

$$\mathcal{H}(u) = \sum_{\text{pairings } / \text{ unpaired}} \prod_{l} \left(1 + u^{-1} h_3 \partial_{z_l}\right) \prod_{\text{pairs } j<k} \left(-\frac{h_1 h_2}{u^2} \frac{1}{4 \sinh^2 \left(\frac{z_j - z_k}{2}\right)} \right)$$

(Feynman, Polychronakos)

- moreover these are guaranteed to satisfy the same commutation relations $\rightsquigarrow \mathcal{W}_\infty$ action on orbital DOF
- the Hamiltonians can be easily generalized to elliptic case, what about ladder operators?

Lessons, questions

- non-linearity is to be expected and probably cannot be avoided (but quadraticity? Miura, Yangian, RTT)
- other algebras with similarly nice properties? Grassmannian $\mathcal{W}$? higher spin square?
- Yangian description of the orthosymplectic case? exceptional subalgebras?
- many things can be generalized to include matrix DOF (rectangular $\mathcal{W}$, spin Calogero models)
- growth of states like of 3d theory, global subalgebra diffeomorphisms of S^2, any connection to membrane?
- classical KP hierarchy has very well developed theory, what are the translations of objects there to the quantum language? tau functions, Hirota equations, Y-systems, T-systems etc.
- BPS state counting, topological vertex, mirror symmetry, spectral curves, topological recursion

Thank you!