Bilocal effective action for tensor models

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Large-N limit

The large-N limit is a precious tool for accessing non-perturbative phenomena in QFT

Typical case: consider a theory with fields transforming in a representation
of a compact Lie group (e.g. $O(N)$ or $U(N)$),
and an action invariant under the (global) group transformations

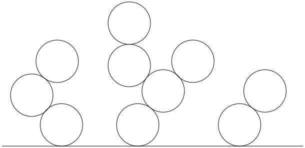
General idea: choose a rescaling of the couplings with N such that

1. the large-N limit of the theory exists and it is non-trivial;
2. only a subset of the Feynman diagrams survives in the limit.

Large-N limit of vectors

e.g. fields in the fundamental representation of $O(N)$ ("$O(N)$ model")

- Large N: **Cactus diagrams**



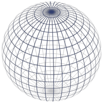
$\rightarrow$ Closed Schwinger-Dyson equation for 2-point function
$=$ mass gap equation (no anomalous dimension)

Large-N limit of matrices

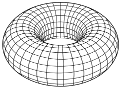
e.g. fields in the adjoint representation of $U(N)$ (Hermitian matrix model)

$\Rightarrow$ genus expansion:

$$\ln Z = \sum_{g \geq 0} N^{2-2g} F_g(\lambda) \sim N^2$$



$$+$$



$$+ O(N^{-2})$$

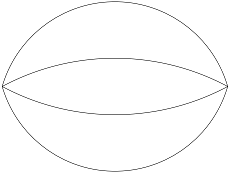
- Large-N limit: **planar diagrams**
  $\rightarrow$ No closed Schwinger-Dyson equation; still very difficult

  In zero dimension there are many techniques for solving matrix models, but they typically become very hard in higher dimensions

Large-N limit of tensors

e.g. fields in the fundamental representation of $O(N)^3$

A new type of large-N limit: **the melonic limit** of tensor-valued field theories



More complicated than the vector case, but simpler than the matrix case

It is a recent discovery [2010-on: Gurau, Rivasseau, Bonzom, Carrozza, Tanasa, ...]

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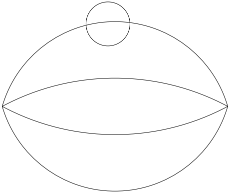

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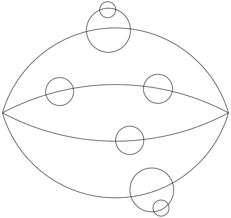

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Melonic revival

Recently, the melonic limit has been rediscovered in the context of AdS/CFT:

- In the SYK model and in models of tensor quantum mechanics, the melonic limit leads to interesting features for a holographic description of extremal black holes
- However, SYK and tensor models have important differences

This talk:

In the SYK model, a bilocal action formulation plays a key role, but an analog formulation was missing for its tensorial cousins

$\Rightarrow$ Introduce the two-particle irreducible (2PI) effective action for tensor models

Overview

- SYK model
- Tensor models in $d = 1$
- 2PI effective action

SYK model

The Sachdev-Ye-Kitaev (SYK) model

A model of N Majorana fermions in $d = 1$: [Sachdev,Ye (1992); Kitaev (2015)]

$$\mathbf{S}_{\mathrm{SYK}}[\psi] = \int dt \left(\frac{1}{2} \sum_{a=1}^{N} \psi_a \partial_t \psi_a + \frac{\mathrm{i}^{q/2}}{q!} \sum_{a_1,\dots,a_q} J_{a_1\dots a_q} \psi_{a_1} \dots \psi_{a_q} \right)$$

where $J_{a_1\dots a_q}$ is a random tensorial coupling ($\Rightarrow$ no $O(N)$ invariance), with Gaussian distribution:

$$P[J_{a_1\dots a_q}] \propto \exp\left\{ -\frac{N^{q-1}(J_{a_1\dots a_q})^2}{2(q-1)!J^2} \right\}$$

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Randomness $\Rightarrow$ **quenched** average of intensive quantities,
e.g. the free energy:

$$\overline{F} = -\frac{1}{N}\overline{\ln Z} = -\frac{1}{N} \int \prod_{a_1<a_2<\dots<a_q} [dJ_{a_1\dots a_q}] P[J_{a_1\dots a_q}] \ln \int [d\psi] e^{-\mathbf{S}_{\mathrm{SYK}}[\psi]}$$

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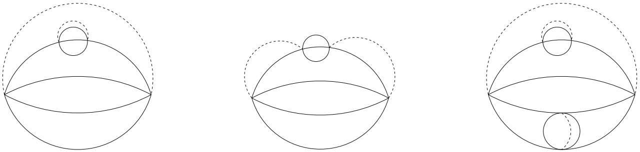
Quenched average can be represented with new lines in connected fermionic graphs



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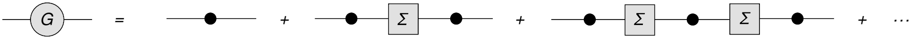
Quenched average can be represented with new lines in connected fermionic graphs

At large-N, after the quenched average on $J_{a_1\dots a_q}$, the leading-order diagrams are melonic

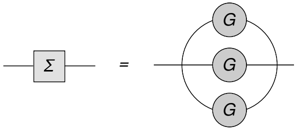
Conformal limit

Because of the melonic dominance, the large-N Schwinger-Dyson equations form a closed equation for the 2-point function, e.g. for $q = 4$:

$$G(\omega) = (-i\omega - \Sigma(\omega))^{-1}$$



$$\Sigma(\omega) = J^2 \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} G(\omega_1) G(\omega_2) G(\omega - \omega_1 - \omega_2)$$



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- The Schwinger-Dyson equations are the field equations for an effective bilocal action

Replica trick

1. A standard method for dealing with quenched average: replica trick, i.e.

$$
\begin{aligned}
\overline{\ln Z} &= \lim_{n\to 0} \partial_n \overline{Z^n} \\
&= \lim_{n\to 0} \partial_n \int \prod_{a_1<a_2<\ldots<a_q} [dJ_{a_1\ldots a_q}] P[J_{a_1\ldots a_q}] \int \prod_{\alpha=1}^{n} [d\psi^\alpha] e^{-\mathbf{S}_{\mathrm{SYK}}[\psi^\alpha]} \\
&= \lim_{n\to 0} \partial_n \int \left(\prod_{\alpha=1}^{n} [d\psi^\alpha] \right) e^{-\frac{1}{2}\sum_\alpha \int_t \psi_a^\alpha \partial_t \psi_a^\alpha + \frac{\mathrm{i}^q J^2}{2qN^{q-1}} \sum_{\alpha,\beta} \int_t \psi_{a_1}^\alpha \ldots \psi_{a_q}^\alpha \int_{t'} \psi_{a_1}^\beta \ldots \psi_{a_q}^\beta}
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2. Introduce bilocal variables: insert

$$\begin{aligned}
1 &= \int \prod_{\alpha\beta}^{1\ldots n} [dG^{\alpha\beta}] \delta \left(NG^{\alpha\beta}(t,t') - \sum_a \psi_a^\alpha(t) \psi_a^\beta(t') \right) \\
&= \int \prod_{\alpha\beta}^{1\ldots n} [dG^{\alpha\beta}][d\Sigma^{\alpha\beta}]\, e^{-\frac{1}{2}\int_{t,t'} \Sigma^{\alpha\beta}(t,t') \left(NG^{\alpha\beta}(t,t') - \sum_a \psi_a^\alpha(t)\psi_a^\beta(t') \right)}
\end{aligned}$$

inside functional integral, use the constraint in the interaction part ($\sim G^q$), and integrate out the fermions (now Gaussian)

Bilocal effective action

$$\overline{Z^n} = \int \left(\prod_{\alpha\beta}^{1\dots n} [dG^{\alpha\beta}][d\Sigma^{\alpha\beta}] \right) e^{-N\mathbf{S}_{\text{eff}}[G,\Sigma]}$$

where

$$\mathbf{S}_{\text{eff}}[G,\Sigma] = -\frac{1}{2}\widehat{\text{Tr}}\ln(\partial_t - \Sigma) + \frac{1}{2}\sum_{\alpha\beta}^{1\dots n}\int_{t,t'}\left(\Sigma^{\alpha\beta}(t,t')G^{\alpha\beta}(t,t') - \frac{J^2}{q}(G^{\alpha\beta}(t,t'))^q\right)$$

- N-dependence factors in front of the action: large-N limit is given by saddle-point approximation
- For SYK there is no replica symmetry breaking at large N ($T_{\text{RSB}} \sim e^{-\sqrt{N}}$) [Georges, Parcollet, Sachdev - 2001], hence we can use a replica-symmetric and diagonal ansatz:

$$G^{\alpha\beta}(t,t') = \delta^{\alpha\beta}\underline{G}(t,t') + \frac{1}{N^{1/2}}g^{\alpha\beta}(t,t')$$

  with $\underline{G}(t,t')$ solving the saddle point equation ($=$ Schwinger-Dyson equation)
- At leading and subleading order in $\frac{1}{N}$, $n=0$ coincides with $n=1$ (quenched $=$ annealed)

Leading order result:

$$\overline{\ln Z}_{LO} = N\left(\frac{1}{2}\text{Tr}[\ln \underline{G}^{-1}] + \frac{1}{2}\text{Tr}[\partial_t \underline{G}(t,t')] + \frac{J^2}{2q}\int_{t,t'}\underline{G}(t,t')^q\right)$$

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again conformal invariant in the IR

Bilocality and holography

SYK has attracted attention because the CFT_1 is expected to have an AdS_2 dual, and thus describe the microscopics of near extremal black holes

Bilocal action formulation is a useful tool in the construction of the holographic dual: bilocality $\sim$ extra dimensions

Basic idea: $2d$ coordinates $\Rightarrow$ $d+1$ coordinates + spin [Das, Jevicki - '03]

In SYK: [Jevicki, Suzuki, Yoon - '16]
$d = 1 \Rightarrow$ no spin

$$t = \frac{1}{2}(t_1 + t_2)\,, \quad z = \frac{1}{2}(t_1 - t_2)$$

$$g(t_1, t_2) \quad \Rightarrow \quad \phi(t, z)$$

From the quadratic fluctuations around saddle point:

$$S_{\text{eff}} \sim \frac{1}{2}\sum_n \int d^2x\sqrt{-g}[-g^{\mu\nu}\partial_\mu\phi_n\partial_\nu\phi_n - h_n(1-h_n)\phi_n^2]$$

$\Rightarrow$ the bilocal field packs a sequence of AdS_2 scalars, with growing mass

$$h_n \simeq 2\Delta + 2n + 1 : \text{ conformal dimension of } \psi_a\partial_t^{2n+1}\psi_a$$

Bilocal action and Jackiw-Teitelboim gravity

Bilocal action has also played an important role in identifying the gravitational part of the bulk dual: [Kitaev - '15; Maldacena,Stanford - '16]

- Fluctuations $g^{\alpha\beta}(t,t')$ around saddle point have an exact zero mode in the IR/conformal limit:

$$\overline{\ln Z}_{NLO} = -\frac{1}{2}\mathrm{Tr}[\ln(I_- - \tilde{\mathcal{K}} I_-)]$$

with the ladder kernel:

$$\tilde{\mathcal{K}}(t_1,t_2;t_3,t_4) = -J^2(q-1)|\underline{G}(t_1,t_2)|^{\frac{q-2}{2}}\,\underline{G}(t_1,t_3)\underline{G}(t_2,t_4)\,|\underline{G}(t_3,t_4)|^{\frac{q-2}{2}}$$

Zero mode: $\tilde{\mathcal{K}} = 1$ eigenmode

$\Rightarrow$ non-conformal corrections lead to an effective action for the zero mode, a.k.a. the Schwarzian action

$$S_{\mathrm{Sch}} = -N\frac{\alpha}{J}\int d\tau \left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)$$

Effective action for pseudo-Golstone modes of reprametrization invariance ($\tau \to f(\tau)$)

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- Jackiw-Teitelboim gravity

$$S_{JT} = -\frac{1}{16\pi G}\left[\int_M d^2x\sqrt{g}(R-2\Lambda)\phi + 2\int_{\partial M} dt\phi K\right]$$

has similar phenomenon: singularity at conformal AdS boundary
$\rightarrow$ cut out near-boundary region
$\rightarrow$ Schwarzian action from on-shell evaluation of the boundary action

$\Rightarrow$ Same pattern of symmetry breaking, controlled by the same effective action

Tensor models in $d = 1$

Tensor models

The same melonic limit, and hence conformal symmetry in the IR, can be obtained without disorder, but replacing vectors with tensors
[Witten - '16]

Three types of models:

- **Colored tensor models**: with q distinguished fields with $q-1$ distinguished indices
  $\Rightarrow$ symmetry group: $O(N)^{q(q-1)/2}$
  [Gurau - '10]
- **Uncolored tensor models**: with only one field with $q-1$ distinguished indices
  $\Rightarrow$ symmetry group: $O(N)^{q-1}$
  [Bonzom, Gurau, Rivasseau - '12 ($U(N)$); Carrozza, Tanasa - '15 ($O(N)$)]
- **Symmetric tensor models**: with only one field with $q-1$ indices, in an irreducible representation of $O(N)$ (e.g. symmetric-traceless, antisymmetric, etc.)
  [Klebanov,Tarnopolsky - '17; DB, Carrozza, Gurau, Kolanowski - '17]

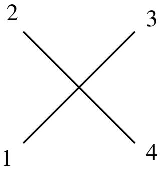
Gurau-Witten model [Gurau - '10; Witten - '16]

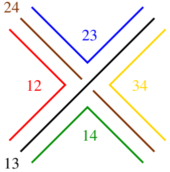
$d = 1$ fermionic generalization of colored tensor model

Fields: q Majorana fermions in fundamental representation of $O(N)^{q-1}$;
field c $(= 1, \dots, q)$ has an index $c_1 (\neq c)$ transforming with the same group element as the index c of field c_1
$\Rightarrow$ symmetry group: $O(N)^{q(q-1)/2}$

$$\begin{aligned}\mathbf{S}_{\mathrm{GW}}[\psi] =& \frac{1}{2} \sum_{c=1}^{q} \int_t \psi^{(c)}_{\mathbf{a_c}}(t) \partial_t \psi^{(c)}_{\mathbf{a_c}}(t) \\ &+ \frac{\mathrm{i}^{q/2}\,\lambda}{N^{(q-1)(q-2)/4}} \int_t \prod_{c=1}^{q} \psi^{(c)}_{\mathbf{a_c}}(t) \prod_{c_1 < c_2} \delta_{a^{c_1}_{c_2} a^{c_2}_{c_1}}\end{aligned}$$

where $\mathbf{a_c} = (a^c_{c_1} | c_1 \in \{1, \dots, q\} \backslash \{c\})$





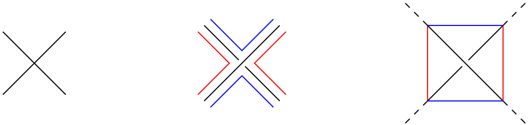
CTKT model

[Carrozza,Tanasa - '15; Klebanov,Tarnopolsky - '17]

$d = 1$ fermionic generalization of uncolored tensor model

Field: Majorana fermion in fundamental representation of $O(N)^{q-1}$.

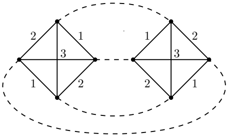
$$\mathbf{S}_{\rm CTKT}[\psi] = \int_t \left(\frac{1}{2} \psi_{abc}(t) \partial_t \psi_{abc}(t) + \frac{\lambda}{4N^{3/2}} \psi_{a_1 a_2 a_3}(t) \psi_{a_1 b_2 b_3}(t) \psi_{b_1 a_2 b_3}(t) \psi_{b_1 b_2 a_3}(t) \right)$$



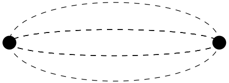
Feynman graphs

Perturbative expansion:

- Represent Wick contraction of two tensors by dashed line, e.g.:



- Ordinary Feynman diagrams, tracking only ordinary spacetime/momentum integrals are obtained by shrinking interaction bubbles to a point:



- The leading order in $1/N$ is given by melonic diagrams
  $\Rightarrow$ same Schwinger-Dyson equation as in SYK

Tensor models vs SYK

Advantages of tensor models:

- No quenched disorder, which is unnatural for AdS/CFT
- In SYK the $O(N)$ symmetry only emerges after quenching;
  in tensor models it is there from the beginning, so it can be gauged:
  $\Rightarrow$ Gauging gets rid of the non-singlet states,
  which are an obstacle in the search for the gravity dual
- Subleading corrections better understood (several years of results from tensor models and no tricky issues with replica limit)

Important differences:

- In non-gauged version, global symmetry becomes almost local in IR: new soft modes besides the Schwarzian mode [Minwalla et al. - '17; DB, Gurau - '18]
- Many more invariants (singlets) in tensor models: complicated bulk dual
- No collective field formulation

2PI effective action for tensor models

JHEP 05 (2018) 156 [arXiv:1802.05500], with R. Gurau

2PI effective action – general definition [Cornwall, Jackiw, Tomboulis - '74]

Define the generating functional:

$$\mathbf{W}[j,k] = \ln \int [d\varphi]\ \exp\left\{ -\mathbf{S}[\varphi] + j_{\mathbf{a}}\varphi_{\mathbf{a}} + \frac{1}{2}\varphi_{\mathbf{a}} k_{\mathbf{ab}} \varphi_{\mathbf{b}} \right\}$$

The 2PI effective action is the double Legendre transform with respect to the sources:

$$\mathbf{\Gamma}[\phi, G] = -\mathbf{W}[\mathbf{J},\mathbf{K}] + \mathbf{J}_{\mathbf{a}}\phi_{\mathbf{a}} + \frac{1}{2}\phi_{\mathbf{a}}\mathbf{K}_{\mathbf{ab}}\phi_{\mathbf{b}} + \frac{1}{2}\mathrm{Tr}[G\mathbf{K}]$$

with $\mathbf{J}$, and $\mathbf{K}$ such that:

$$\frac{\delta \mathbf{W}}{\delta j_{\mathbf{a}}}[\mathbf{J},\mathbf{K}] = \langle \varphi_{\mathbf{a}} \rangle_{\mathbf{J},\mathbf{K}} = \phi_{\mathbf{a}}$$

$$\frac{\delta \mathbf{W}}{\delta k_{\mathbf{ab}}}[\mathbf{J},\mathbf{K}] = \frac{1}{2}\langle \varphi_{\mathbf{a}}\varphi_{\mathbf{b}} \rangle_{\mathbf{J},\mathbf{K}} = \frac{1}{2}\left(G_{\mathbf{ab}} + \phi_{\mathbf{a}}\phi_{\mathbf{b}}\right)$$

and

$$\frac{\delta \mathbf{\Gamma}}{\delta \phi_{\mathbf{a}}}[\phi, G] = \mathbf{J}_{\mathbf{a}}[\phi, G] + \mathbf{K}_{\mathbf{ab}}[\phi, G]\phi_{\mathbf{b}}$$

$$\frac{\delta \mathbf{\Gamma}}{\delta G_{\mathbf{ab}}}[\phi, G] = \frac{1}{2}\mathbf{K}_{\mathbf{ba}}[\phi, G]$$

$\mathbf{J} = \mathbf{K} = 0 \ \Rightarrow\ \underline{\phi} = \langle \varphi \rangle,\ \underline{G} = 2\langle \varphi\varphi \rangle_c$

2PI effective action – loop expansion [Cornwall, Jackiw, Tomboulis - '74]

It is not hard to prove that

$$\mathbf{\Gamma}[\phi, G] = \underbrace{\mathbf{S}[\phi]}_{\text{tree level}} + \underbrace{\frac{1}{2}\mathrm{Tr}[\ln G^{-1}] + \frac{1}{2}\mathrm{Tr}[\mathbf{S}_{\phi\phi}[\phi]G]}_{\text{one loop}} + \underbrace{\mathbf{\Gamma}_2[\phi, G]}_{\text{two or more loops}}$$

with the following Feynman rules:

vertices:	$\mathbf{S}_{\mathrm{int}}[\phi, \varphi] = \mathbf{S}[\phi + \varphi]_{\text{starting at cubic order in } \varphi}$
propagator:	$G(x, y)$

2PI effective action – loop expansion

[Cornwall, Jackiw, Tomboulis - '74]

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propagator: $G(x, y)$

$\mathbf{\Gamma}_2[\phi, G]$ is given by the sum of all the $(n \geq 2)$-loops **two-particle irreducible** vacuum graphs

Hint:

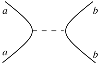
$$\frac{\delta \mathbf{\Gamma}}{\delta G_{\mathbf{ab}}} = 0 \quad \Rightarrow \quad G^{-1} = \mathbf{S}_{\phi\phi}[\phi] + 2\frac{\delta \mathbf{\Gamma}_2}{\delta G} = G_0^{-1} - \Sigma$$

Large-N limit of the 2PI effective action

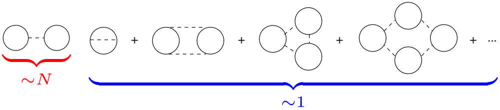
– vector $O(N)$ model –

$$\mathbf{S}[\varphi] = \frac{1}{2}\int_{x,y} \varphi_a(x) C^{-1}(x,y)\varphi_a(y) + \frac{\lambda}{4!N}\int_x (\varphi_a(x)\varphi_a(x))^2$$

vertex $(\phi = 0)$:



2PI vacuum graphs:



$+ O\left(\frac{1}{N}\right)$

$$\mathbf{\Gamma}_2[\phi, G] = N\frac{\lambda}{4!}\int_x G(x,x)^2 + \frac{1}{2}\mathrm{Tr}[\ln(\mathbf{1} + \frac{\lambda}{6}G(x,y)^2)] + O\left(\frac{1}{N}\right)$$

Large-N limit of the 2PI effective action

– vector $O(N)$ model –

$$\mathbf{S}[\varphi] = \frac{1}{2}\int_{x,y} \varphi_a(x) C^{-1}(x,y)\varphi_a(y) + \frac{\lambda}{4!N}\int_x (\varphi_a(x)\varphi_a(x))^2$$

Loop structure of $1/N$ expansion: from same trick as in SYK

$$\begin{aligned} Z &= \int [d\varphi]\, e^{-\mathbf{S}[\varphi]} \\ &= \int [d\varphi][d\tilde{G}][d\tilde{\Sigma}]\, e^{-\mathbf{S}[\varphi] - \frac{1}{2}\int_{x,y}\tilde{\Sigma}(x,y)\left(N\tilde{G}(x,y) - \sum_a \varphi_a(x)\varphi_a(y)\right)} \\ &= \int [d\tilde{G}][d\tilde{\Sigma}]\, e^{-N\left\{\frac{1}{2}\mathrm{Tr}[(C^{-1}-\tilde{\Sigma})\tilde{G}] + \frac{1}{2}\mathrm{Tr}[\ln(\tilde{\Sigma})] + \frac{\lambda}{4!}\int_x \tilde{G}(x,x)^2\right\}} \\ &\equiv \int [d\tilde{G}][d\tilde{\Sigma}]\, e^{-N\mathbf{S}_{\mathrm{eff}}[\tilde{G},\tilde{\Sigma}]} \end{aligned}$$

Large-N limit of the 2PI effective action

– GW model –

$$\mathbf{S}_{\mathrm{GW}}[\psi] = \frac{1}{2}\sum_{c=1}^{q}\int_t \psi_{\mathbf{a_c}}^{(c)}(t)\partial_t\psi_{\mathbf{a_c}}^{(c)}(t) + \frac{\mathrm{i}^{q/2}\,\lambda}{N^{(q-1)(q-2)/4}}\int_t \prod_{c=1}^{q}\psi_{\mathbf{a_c}}^{(c)}(t)\prod_{c_1<c_2}\delta_{a_{c_1c_2}a_{c_2c_1}}$$

Introduce a bilocal source for each color and perform the double Legendre transform:

$$\mathbf{\Gamma}[\Psi_{\mathbf{a_c}}^{(c)}(t), G_{\mathbf{a_c}\mathbf{b_c}}^{(c)}(t,t')] = \mathbf{S}_{\mathrm{GW}}[\Psi^{(c)}] + \frac{1}{2}\sum_{c=1}^{q}\mathrm{Tr}[\ln(G^{(c)})] - \frac{1}{2}\sum_{c=1}^{q}\mathrm{Tr}[(G_0^{(c)})^{-1}G^{(c)}] + \mathbf{\Gamma}_2[\Psi^{(c)}, G^{(c)}]$$

with

$$\langle\psi_{\mathbf{a_c}}^{(c)}(t)\rangle = \underline{\Psi}_{\mathbf{a_c}}^{(c)}(t)$$

$$\langle\Psi_{\mathbf{a_c}}^{(c)}(t)\Psi_{\mathbf{b_c}}^{(c)}(t')\rangle = \frac{1}{2}\left(\underline{G}_{\mathbf{a_c}\mathbf{b_c}}^{(c)}(t,t') + \underline{\Psi}_{\mathbf{a_c}}^{(c)}(t)\underline{\Psi}_{\mathbf{b_c}}^{(c)}(t')\right)$$

Large-N limit of the 2PI effective action

– GW model –

$$\mathbf{S}_{\rm GW}[\psi] = \frac{1}{2}\sum_{c=1}^{q}\int_t \psi^{(c)}_{\mathbf{a_c}}(t)\partial_t\psi^{(c)}_{\mathbf{a_c}}(t) + \frac{\mathrm{i}^{q/2}\,\lambda}{N^{(q-1)(q-2)/4}}\int_t\prod_{c=1}^{q}\psi^{(c)}_{\mathbf{a_c}}(t)\prod_{c_1<c_2}\delta_{a_{c_1c_2}a_{c_2c_1}}$$

Introduce a bilocal source for each color and perform the double Legendre transform:

$$\begin{aligned}\mathbf{\Gamma}[\Psi^{(c)}_{\mathbf{a_c}}(t), G^{(c)}_{\mathbf{a_c}\mathbf{b_c}}(t,t')] = {} & \mathbf{S}_{\rm GW}[\Psi^{(c)}] + \frac{1}{2}\sum_{c=1}^{q}\mathrm{Tr}[\ln(G^{(c)})] - \frac{1}{2}\sum_{c=1}^{q}\mathrm{Tr}[(G_0^{(c)})^{-1}G^{(c)}] \\ & + \mathbf{\Gamma}_2[\Psi^{(c)}, G^{(c)}]\end{aligned}$$

with

$$\langle\psi^{(c)}_{\mathbf{a_c}}(t)\rangle = \underline{\Psi}^{(c)}_{\mathbf{a_c}}(t)$$

$$\langle\Psi^{(c)}_{\mathbf{a_c}}(t)\Psi^{(c)}_{\mathbf{b_c}}(t')\rangle = \frac{1}{2}\left(\underline{G}^{(c)}_{\mathbf{a_c}\mathbf{b_c}}(t,t') + \underline{\Psi}^{(c)}_{\mathbf{a_c}}(t)\underline{\Psi}^{(c)}_{\mathbf{b_c}}(t')\right)$$

In the symmetric phase

$$\underline{\Psi}^{(c)}_{\mathbf{a_c}}(t) = 0, \quad \underline{G}^{(c)}_{\mathbf{a_c}\mathbf{b_c}}(t,t') = G(t,t')\prod_{c'\neq c}\delta_{a^c_{c'},b^c_{c'}}$$

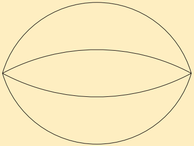
$\Rightarrow$ each trace on a color index counts as a factor N

Leading-order 2PI effective action for GW model

Large N:

Counting traces = counting faces in stranded graph ⇒ **leading order is given by melons**

There is only one melon graph which is also 2PI: the fundamental melon



$$\mathbf{\Gamma}_2^{\mathrm{LO}}[\Psi^{(c)} = 0, G^{(c)} = G] = -\frac{\lambda^2}{2N^{(q-1)(q-2)/2}} \int_{t,t'} \prod_{c=1}^{q} G^{(c)}_{\mathbf{a_c b_c}}(t,t') \prod_{c_1 < c_2} \delta_{a^{c_1}_{c_2} a^{c_2}_{c_1}} \delta_{b^{c_1}_{c_2} b^{c_2}_{c_1}}$$

$$= -\frac{\lambda^2 N^{q-1}}{2} \int_{t,t'} G(t,t')^q$$

⇒ Same result as in the SYK model

Leading-order 2PI effective action for CTKT model

For the CTKT model we obtain a similar result:

$$\mathbf{\Gamma}[0, G] = -\frac{1}{2}\mathrm{Tr}[\ln G^{-1}_{a_1 a_2 a_3 b_1 b_2 b_3}] - \frac{1}{2}\mathrm{Tr}[\partial_t G_{a_1 a_2 a_3 b_1 b_2 b_3}(t, t')] + \mathbf{\Gamma}_2^{(3)}[G]$$

with

$$\mathbf{\Gamma}_2^{(3)}[G] =$$

$$= \frac{-\lambda^2}{8N^3}\int_{t,t'} G_{a_1 a_2 a_3 b_1 b_2 b_3}(t, t') G_{a_1 a_2' a_3' b_1 b_2' b_3'}(t, t') G_{a_1' a_2 a_3' b_1' b_2 b_3'}(t, t') G_{a_1' a_2' a_3 b_1' b_2' b_3}(t, t')$$

$$= -\frac{1}{8}\lambda^2 N^3 \int_{t,t'} G(t, t')^4$$

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with

$$\begin{aligned}
&\mathbf{\Gamma}_2^{(3)}[G] = \\
&= \frac{-\lambda^2}{8N^3}\int_{t,t'} G_{a_1a_2a_3b_1b_2b_3}(t,t')G_{a_1a'_2a'_3b_1b'_2b'_3}(t,t')G_{a'_1a_2a'_3b'_1b_2b'_3}(t,t')G_{a'_1a'_2a_3b'_1b'_2b_3}(t,t') \\
&= -\frac{1}{8}\lambda^2N^3\int_{t,t'} G(t,t')^4
\end{aligned}$$

In the IR limit, global $O(N)^{q-1}$ symmetry becomes a local symmetry: new zero modes

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In the IR limit, global $O(N)^{q-1}$ symmetry becomes a local symmetry: new zero modes

As in the case of the Schwarzian action, reintroducing the derivative term leads to an effective action for the soft modes; this time a non-linear sigma model ($V_c \in O(N)$):

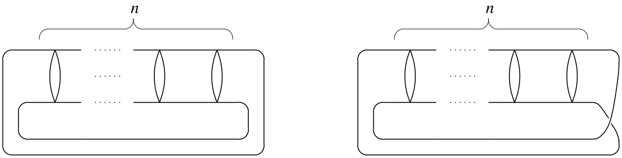
$$S_{\mathrm{eff}} = -\frac{\alpha}{2}N^{q-2}\int_t \sum_{c=1}^{q-1}\mathrm{Tr}\left[\left(V_c^{-1}\partial_t V_c\right)^2\right]$$

Next three subleading orders for GW model

Leading order is $\sim N^{q-1}$, while subleading orders start at $\sim N^2$:

$$\mathbf{\Gamma}_2[G] = \mathbf{\Gamma}_2^{(q-1)}[G] + \mathbf{\Gamma}_2^{(2)}[G] + \mathbf{\Gamma}_2^{(1)}[G] + \mathbf{\Gamma}_2^{(0)}[G] \; ,$$

Subleading diagrams are an infinite family of ladders:



where colors and twist lead to different scalings with N: [Bonzom, Lionni, Tanasa - 2017]

Order N^2	$\Rightarrow$	rails: alternating colors without twist
Order N	$\Rightarrow$	rails: alternating colors with twist
Order 1	$\Rightarrow$	rails: non-alternating colors with or without twist

Resummation of subleading contributions for GW model

Subleading diagrams can be resummed:

$$\boldsymbol{\Gamma}_2^{(2)}[G] = N^2 \frac{1}{4}\binom{q}{2} \mathrm{Tr}\left[I^{=} \ln\left(1 - \lambda^4 \hat{\mathcal{K}}^2\right)\right]$$

$$\boldsymbol{\Gamma}_2^{(1)}[G] = N \frac{1}{4}\binom{q}{2} \mathrm{Tr}\left[(-I^{\times}) \ln\left(1 - \lambda^4 \hat{\mathcal{K}}^2\right)\right]$$

$$\boldsymbol{\Gamma}_2^{(0)}[G] = \frac{1}{2} \mathrm{Tr}\left[I_{-} \ln\left(1 - (q-1)\lambda^2 \hat{\mathcal{K}}\right)\right] + \frac{q-1}{2} \mathrm{Tr}\left[I_{-} \ln\left(1 + \lambda^2 \hat{\mathcal{K}}\right)\right] - \frac{1}{2}\binom{q}{2} \mathrm{Tr}\left[I_{-} \ln\left(1 - \lambda^4 \hat{\mathcal{K}}^2\right)\right]$$

where $\hat{\mathcal{K}}$ is a ladder kernel, and $I_{\pm} = (I^{=} \pm I^{\times})/2$ are projectors on (anti-)symmetric bilocal functions

Free energy

Putting all together we have the free energy (for $q > 4$):

$$
\begin{aligned}
-\ln Z = \mathbf{\Gamma}[0,\underline{G}] =& N^{q-1}\frac{q}{2}\mathrm{Tr}[\ln(\underline{G}^{(0)})] - N^{q-1}\frac{q}{2}\mathrm{Tr}[\partial_t \underline{G}^{(0)}] - N^{q-1}\frac{\lambda^2}{2}\int_{t,t'}\underline{G}^{(0)}(t,t')^q \\
&+ \left[\frac{N(N-1)}{2}\binom{q}{2}\right]\frac{1}{2}\mathrm{Tr}\left[\ln\left(1-\lambda^4[\underline{\hat{\mathcal{K}}}^{(0)}]^2 I_+\right)\right] \\
&+ \left[\left(\frac{N(N-1)}{2}+(N-1)\right)\binom{q}{2}\right]\frac{1}{2}\mathrm{Tr}\left[\ln\left(1-\lambda^4[\underline{\hat{\mathcal{K}}}^{(0)}]^2 I_-\right)\right] \\
&+ (q-1)\frac{1}{2}\mathrm{Tr}\left[\ln\left(1+\lambda^2[\underline{\hat{\mathcal{K}}}^{(0)}] I_-\right)\right] + \frac{1}{2}\mathrm{Tr}\left[\ln\left(1-(q-1)\lambda^2[\underline{\hat{\mathcal{K}}}^{(0)}] I_-\right)\right]
\end{aligned}
$$

where $\underline{G}^{(0)}$ and $\underline{\hat{\mathcal{K}}}^{(0)}$ are the leading-order on-shell two-point function and its ladder kernel

(subleading corrections to $\underline{G}^{(0)}$ contribute the free energy at order N^{5-q})

We have rearranged subleading contributions to highlight a peculiar structure...

Interpretation in terms of an auxiliary theory

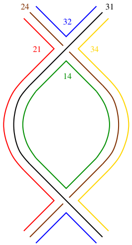
The trace-log terms can be interpreted as the one loop correction to the leading-order action

$$N^{q-1}\frac{q}{2}\mathrm{Tr}[\ln(G)] - N^{q-1}\frac{q}{2}\mathrm{Tr}[\partial_t G] - N^{q-1}\frac{\lambda^2}{2}\int_{t,t'} G(t,t')^q$$

Hint:
Expanding around the saddle point

$$G^{(c)}_{\mathbf{a_c b_c}}(t,t') = \underline{G}^{(0)}(t,t')\delta_{\mathbf{a_c b_c}} + g^{(c)}_{\mathbf{a_c b_c}}(t,t')$$

the one loop correction gives the $\det(\mathbb{I} - \lambda^2\mathbb{K})^{-1/2}$, where the operator $\mathbb{K}$ is a matrix in color space built out of kernels $\mathcal{K}^{(c_1 c_2)}$, such as



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$\Rightarrow$ fluctuations decompose into symmetric traceless matrices, antisymmetric matrices and scalars:

$$g^{(c)}_{\mathbf{a_c}\mathbf{b_c}}(t,t') = g^{(c)}(t,t')\prod_{i\neq c}\delta_{a^c_i b^c_i} + \sum_{i\neq c} g^{(ci)}_{a^c_i b^c_i}(t,t')\prod_{j\neq i,c}\delta_{a^c_j b^c_j} + \hat{g}^{(c)}_{\mathbf{a_c}\mathbf{b_c}}(t,t')$$

and $g^{(ci)}_{a_{ci}b_{ci}}(t,t')$ decomposed in symmetric traceless and antisymmetric parts

Taking into account the matrix structure of $\mathbb{K}$, we find exactly the free energy above

Conclusions and outlook

Conclusions and outlook

- The SYK model has brought the melonic limit of tensor models under the spotlight
- Tensor models offer a number of advantages over the SYK model (in particular because of no disorder)
- They also present some differences and extra challenges (new light modes, many more invariants, ...)
- We advocated the use of the 2PI formalism to bypass the lack of a bilocal reformulation of the path integral, showing that it reproduces the bilocal effective action of SYK
- Large-N of colored tensor model: only one 2PI diagram at leading order; infinite but summable families of 2PI diagrams at first three subleading orders
- Surprisingly, the $1/N$ expansion of the 2PI effective action of the colored tensor model suggests the existence of an effective bilocal reformulation, at least up to order N^0
- Many open directions and questions, in particular concerning the holographic interpretation of tensor models