Holography, Unfolding and Higher-Spin Theories

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HS theory

Higher derivatives in interactions

A.Bengtsson, I.Bengtsson, Brink (1983), Berends, Burgers, van Dam (1984)

$$S = S^2 + S^3 + \dots , \qquad S^3 = \sum_{p,q,r} (D^p\varphi)(D^q\varphi)(D^r\varphi)\rho^{p+q+r+\frac{1}{2}d-3}$$

HS Gauge Theories $(m=0)$: Fradkin, M.V. (1987)

$$AdS_d : \qquad [D_n, D_m] \sim \rho^{-2} = \lambda^2$$

AdS/CFT:

$(3d, m=0) \otimes (3d, m=0) = \sum_{s=0}^{\infty}(4d, m=0)$ Flato, Fronsdal (1978);

Sundborg (2001), Sezgin, Sundell (2002,2003), Klebanov, Polyakov (2002),

Giombi, Yin (2009). . .

Maldacena, Zhiboedov (2011) Thm:

Unitary, conformal, local theory conserved current of spin $s > 2$ is free

Is a boundary dual of AdS_4 HS theory free?

Main results

CFT_3 **dual of** AdS_4 **HS theory: 3d superconformal HS theory**

Holography: Unfolding

Plan

Unfolded dynamics

First-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)) \qquad \text{initial values:} \quad q^i(t_0)$$

$\#$ degrees of freedom $= \#$ of dynamical variables

Field theory: infinite $\#$ of degrees of freedom $=$ spaces of functions$=$

infinite $\#$ of undetermined derivatives (generalized momenta)

Dirac approach is nice and efficient but noncovariant.

Covariant extension $t \to x^n$?

Unfolded dynamics: multidimensional generalization

$$\frac{\partial}{\partial t} \to d\,, \qquad q^i(t) \to W^\Omega(x) = dx^{n_1} \wedge \ldots \wedge dx^{n_p} W^\Omega_{n_1 \ldots n_p}(x)$$

a set of differential forms

Unfolded equations

$$dW^{\Omega}(x) = G^{\Omega}(W(x))\,, \qquad d = dx^n \partial_n$$

$G^{\Omega}(W)$: **function of "supercoordinates"** W^{α}

$$G^{\Omega}(W) = \sum_{n=1}^{\infty} f^{\Omega}{}_{\Lambda_1 \ldots \Lambda_n} W^{\Lambda_1} \wedge \ldots \wedge W^{\Lambda_n}$$

$d > 1$: **Nontrivial compatibility conditions**

$$G^{\Lambda}(W) \wedge \frac{\partial G^{\Omega}(W)}{\partial W^{\Lambda}} \equiv 0$$

Any solution to generalized Jacobi identities: FDA

Sullivan (1968); D'Auria and Fre (1982)

The unfolded equation is invariant under the gauge transformation

$$\delta W^{\Omega} = d\varepsilon^{\Omega} + \varepsilon^{\Lambda} \frac{\partial G^{\Omega}(W)}{\partial W^{\Lambda}}\,,$$

where the gauge parameter $\varepsilon^{\Omega}(x)$ **is a** $(p_{\Omega} - 1)$**-form.**

(No gauge parameters for 0-forms W^{Ω}**)**

Vacuum geometry

h: **a Lie algebra.** $\omega = \omega^\alpha T_\alpha$: **a 1-form taking values in** h.

$$G(\omega) = -\omega \wedge \omega \equiv -\frac{1}{2}\omega^\alpha \wedge \omega^\beta [T_\alpha, T_\beta]$$

the unfolded equation with $W = \omega$ **has the zero-curvature form**

$$d\omega + \omega \wedge \omega = 0 \,.$$

Compatibility condition: Jacobi identity for h.

The FDA gauge transformation is the usual gauge transformation of the connection ω.

The zero-curvature equations: background geometry in a coordinate independent way.

If h **is Poincare or anti-de Sitter algebra it describes Minkowski or** AdS_d **space-time**

Free fields unfolded

Let W^{Ω} contain p-forms $\mathcal{C}^i$ (e.g. 0-forms) and G^i be linear in ω and C

$$G^i = -\omega^{\alpha}(T_{\alpha})^i{}_j \wedge \mathcal{C}^j \,.$$

The compatibility condition implies that $(T_{\alpha})^i{}_j$ form some representation T of h, acting in a carrier space V of $\mathcal{C}^i$. The unfolded equation is

$$D_{\omega}\mathcal{C} = 0$$

$D_{\omega} \equiv d + \omega$: covariant derivative in the h-module V.

Covariant constancy equation : linear equations in a chosen background

h: global symmetry

Scalar field example

$s=0$: **infinite set of totally symmetric** 0**-forms** $C_{m_1\dots m_n}(x)$ $(n=0,1,2,\dots)$.

Off-shell unfolded equations

$$dC_{m_1\dots m_n} = e^k C_{m_1\dots m_n k} \qquad (n=0,1,\dots)\,,$$

Cartesian coordinates: $D^L = d$. **The space** V **of** $C_{m_1\dots m_n}$ **forms an (infinite dimensional)** $iso(d-1,1)$**–module.**

First two equations

$$\partial_n C = C_n\,, \qquad \partial_n C_m = C_{mn}$$

All other equations express highest tensors in terms of higher-order derivatives

$$C_{m_1\dots m_n} = \partial_{m_1}\dots\partial_{m_n} C\,.$$

$C_{n_1\dots n_n}$ **describe all derivatives of** $C(x)$. **The system is off-shell: it is equivalent to an infinite set of constraints**

On-shell system: $C^k{}_{k m_3\dots m_n}(x) = 0$

Invariant functionals via Q–cohomology

Equivalent form of compatibility condition

$$Q^2 = 0\,, \qquad Q = G^{\Omega}(W)\frac{\partial}{\partial W^{\Omega}}$$

Q-manifolds

Hamiltonian-like form of the unfolded equations

$$dF(W(x)) = Q(F(W(x))\,, \qquad \forall F(W)\,.$$

Invariant functionals

$$S = \int L(W(x))\,, \qquad QL = 0 \qquad \textbf{(2005)}$$

$L = QM$ **:** **total derivatives**

Actions and conserved charges: Q cohomology

for off-shell and on-shell unfolded systems, respectively

Properties

- General applicability
- Manifest (HS) gauge invariance
- Invariance under diffeomorphisms

  Exterior algebra formalism
- Interactions: nonlinear deformation of $G^{\Omega}(W)$
- Local degrees of freedom are in 0-forms $C^i(x_0)$ at any $x = x_0$ (as $q(t_0)$)

  infinite-dimensional module dual to the space of single-particle states
- Independence of ambient space-time

  Geometry is encoded by $G^{\Omega}(W)$

Unfolding and holographic duality

Unfolded formulation unifies various dual versions of the same system.

Duality in the same space-time: ambiguity in what is chosen to be dynamical or auxiliary fields.

Holographic duality between theories in different dimensions: universal unfolded system admits different space-time interpretations.

Extension of space-time without changing dynamics by letting the differential d and differential forms W to live in a larger space

$$d = dX^n \frac{\partial}{\partial X^n} \to \tilde{d} = dX^n \frac{\partial}{\partial X^n} + d\widehat{X}^{\widehat{n}} \frac{\partial}{\partial \widehat{X}^{\widehat{n}}}\,, \qquad dX^n W_n \to dX^n W_n + d\widehat{X}^{\widehat{n}} \widehat{W}_{\widehat{n}}\,,$$

$\widehat{X}^{\widehat{n}}$ are some additional coordinates.

$$\tilde{d} W^{\Omega}(X, \widehat{X}) = G^{\Omega}(W(X, \widehat{X}))$$

A particular space-time interpretation of a universal unfolded system, e.g, whether a system is on-shell or off-shell, depends not only on $G^{\Omega}(W)$ but, in the first place, on a space-time M^d and chosen vacuum solution $W_0(X)$.

Two unfolded systems in different space-times are equivalent (dual) if they have the same unfolded form.

Most direct way to establish holographic duality between two theories: unfold both to see whether the operators Q of their unfolded formulations coincide.

Given unfolded system generates a class of holographically dual theories in different dimensions.

HS gauge connections in AdS_4

Gauge 1-forms $\omega_{\alpha_1 \ldots \alpha_n, \dot\beta_1 \ldots \dot\beta_m}, \qquad n+m = 2(s-1)$

$s=1: \quad \omega(x) = dx^n \omega_n(x)$

$s=2: \quad \omega_{\alpha\dot\beta}(x), \qquad \omega_{\alpha\beta}(x), \qquad \bar\omega_{\dot\alpha\dot\beta}(x)$

$s=3/2: \quad \omega_\alpha(x), \qquad \bar\omega_{\dot\alpha}(x)$

Frame-like fields: $|n-m| = 0$ **(bosons) or** $|n-m| = 1$ **fermions**

Auxiliary Lorentz-like fields: $|n-m| = 2$ **(bosons)**

Extra fields: $|n-m| > 2$

1987

Gauge invariant field strengths

0-forms $\qquad C_{\alpha_1\dots\alpha_n\,,\dot\beta_1\dots\dot\beta_m}\,, \qquad |n-m| = 2s$

(Anti)selfdual Weyl tensors carry only (dotted)undotted spinor indices

$s = 0: \quad C(x)$

$s = 1/2: \quad C_\alpha(x)\,, \qquad \bar C_{\dot\alpha}(x)$

$s = 1: \quad C_{\alpha\beta}\,, \qquad \bar C_{\dot\alpha\dot\beta}$

$s = 3/2: \quad C_{\alpha\beta\gamma}\,, \qquad \bar C_{\dot\alpha\dot\beta\dot\gamma}$

$s = 2: \quad C_{\alpha_1\dots\alpha_4}\,, \qquad \bar C_{\dot\alpha_1\dots\dot\alpha_4}$

Formulae simplify in terms of generating functions $\omega(y,\bar y\mid x)$, $C(y,\bar y\mid x)$

$$A(y,\bar y\mid x) = i\sum_{n,m=0}^{\infty}\frac{1}{n!m!}y_{\alpha_1}\dots y_{\alpha_n}\bar y_{\dot\beta_1}\dots\bar y_{\dot\beta_m}A^{\alpha_1\dots\alpha_n}{}_{,}{}^{\dot\beta_1\dots\dot\beta_m}(x)$$

Traceless tensors by virtue of Penrose formula:

$$p^{\alpha\dot\beta} = y^\alpha\bar y^{\dot\beta} \quad\Rightarrow\quad p^{\alpha\dot\beta}p_{\alpha\dot\beta} = 0 \quad\Leftrightarrow\quad p^n p_n = 0\,.$$

Twistor auxiliary variables y^α, $\bar y^{\dot\alpha}$ **put the system on-shell**

Central on-shell theorem

Infinite set of spins $s = 0, 1/2, 1, 3/2, 2\dots$

Fermions require doubling of fields

$$\omega^{ii}(y,\bar y \mid x)\,, \qquad C^{i1-i}(y,\bar y \mid x)\,, \qquad i = 0,1\,,$$

$$\bar\omega^{ii}(y,\bar y \mid x) = \omega^{ii}(\bar y, y \mid x)\,, \qquad \bar C^{i\,1-i}(y,\bar y \mid x) = C^{1-i\,i}(\bar y, y \mid x)\,.$$

The full unfolded system for the doubled sets of free fields is

- ⋆ $$R_1^{ii}(y,\overline{y} \mid x) = \eta\,\overline{H}^{\dot\alpha\dot\beta}\frac{\partial^2}{\partial\overline{y}^{\dot\alpha}\partial\overline{y}^{\dot\beta}}\;C^{1-i\,i}(0,\overline{y} \mid x) + \bar\eta\,H^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha\partial y^\beta}\;C^{i\,1-i}(y,0 \mid x)$$
- ⋆ $$\tilde D_0 C^{i\,1-i}(y,\overline{y} \mid x) = 0$$

$$R_1(y,\bar y \mid x) = D_0^{ad}\omega(y,\bar y \mid x) \qquad H^{\alpha\beta} = e^\alpha{}_{\dot\alpha}\wedge e^{\beta\dot\alpha}\,, \quad \overline{H}^{\dot\alpha\dot\beta} = e_\alpha{}^{\dot\alpha}\wedge e^{\alpha\dot\beta}\,,$$

$$D_0^{ad}\omega = D^L - \lambda e^{\alpha\dot\beta}\left(y_\alpha\frac{\partial}{\partial\bar y^{\dot\beta}} + \frac{\partial}{\partial y^\alpha}\bar y_{\dot\beta}\right)\,, \qquad \tilde D_0 = D^L + \lambda e^{\alpha\dot\beta}\left(y_\alpha\bar y_{\dot\beta} + \frac{\partial^2}{\partial y^\alpha\partial\bar y^{\dot\beta}}\right)\,,$$

$$D^L A = d_x - \left(\omega^{\alpha\beta}y_\alpha\frac{\partial}{\partial y^\beta} + \bar\omega^{\dot\alpha\dot\beta}\bar y_{\dot\alpha}\frac{\partial}{\partial\bar y^{\dot\beta}}\right)\,.$$

Non-Abelian HS algebra

Star product

$$(f * g)(Y) = \int dSdT f(Y+S)g(Y+T) \exp -iS_A T^A$$

$$[Y_A, Y_B]_* = 2iC_{AB}\,, \qquad C_{\alpha\beta} = \epsilon_{\alpha\beta}\,, \qquad C_{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\beta}$$

Non-Abelian HS curvature

$$R_1(y,\bar y|x) \to R(y,\bar y|x) = d\omega(y,\bar y|x) + \omega(y,\bar y|x) * \omega(y,\bar y|x)$$

$$\tilde D_0 C(y,\bar y|x) \to \tilde D C(y,\bar y|x) = dC(y,\bar y|x) + \omega(y,\bar y|x) * C(y,\bar y|x) - C(y,\bar y|x) * \omega(y,-\bar y|x)$$

Unfolding as twistor transform

Twistor transform

$$\begin{tikzcd} & \mathbf{C}(Y|x) \arrow[dl, "\eta"'] \arrow[dr, "\nu"] & \\ \mathbf{M}(x) & & \mathbf{T}(Y)\,. \end{tikzcd}$$

$W^{\Omega}(Y|x)$ **are functions on the "correspondence space"** C.

Space-time $\mathbf{M}$: **coordinates** x. **Twistor space** $\mathbf{T}$: **coordinates** Y.

Unfolded equations describe the Penrose transform by mapping functions on $\mathbf{T}$ **to solutions of field equations in** $\mathbf{M}$.

Being simple in terms of unfolded dynamics and the corresponding twistor space $\mathbf{T}$, **holographic duality in terms of usual space-time may be complicated requiring solution of at least one of the two unfolded systems: a nontrivial nonlinear integral map.**

$Sp(2M)$ invariant equations

Conformal invariant massless equations in $d = 3,4,6$: $Sp(2M)$ invariant unfolded equations Bandos, Lukierski, Sorokin (1999); MV (2002) Bandos, Bekaert, de Azcarraga, Sorokin, Tsulaia (2005)

$$dX^{AB}\left(\frac{\partial}{\partial X^{AB}} \pm \frac{\partial^2}{\partial Y^A \partial Y^B}\right)C(Y|X) = 0\,, \qquad A, B = 1, \dots M\,.$$

$M = 2$: $3d$ **massless fields:** $Sp(4)$ **is** $3d$ **conformal group** Shaynkman, MV (2001)

$M = 4$: $Sp(8)$ **extends** $4d$ **conformal group** $su(2,2)$.

Rank r unfolded equations in $\mathcal{M}_M$ from tensoring of Fock modules

Gelfond, MV (2003)

$$dX^{AB}\left(\frac{\partial}{\partial X^{AB}} + \eta_{ij}\frac{\partial^2}{\partial Y_i^A \partial Y_j^B}\right)C(Y|X) = 0\,, \qquad i, j = 1, \dots r\,, A, B = 1, \dots M\,.$$

For diagonal η^{ij} higher-rank equations are satisfied by products of rank–one fields

$$C(Y_i|X) = C_1(Y_1|X)C_2(Y_2|X)\dots C_r(Y_r|X)\,, \qquad D^{tw}C(Y|x) = 0\,.$$

Higher rank as higher dimension

A rank–r field in $\mathcal{M}_M \sim$ a rank–one field in $\mathcal{M}_{rM}$ with coordinates X^{AB}_{ij}.

$$Y^A_i \to Y^{\widetilde{A}}, \qquad \widetilde{A} = 1 \ldots rM$$

Embedding of $\mathcal{M}_M$ into $\mathcal{M}_{rM}$

$$X^{AB}_{11} = X^{AB}_{22} = \ldots = X^{AB}_{rr} = X^{AB}$$

$3d$ conformal currents:

a rank-two field in $\mathcal{M}_2$ $(d = 3) \sim$ rank-one field in $\mathcal{M}_4$ $(d = 4)$.

A single rank-one field in $\mathcal{M}_4$ describes all $4d$ conformal fields.

Realization of Flato-Fronsdal Thm

Rank-two equations and conserved currents

The rank-two equation can be rewritten in the form

$$\left\{ \frac{\partial}{\partial X^{AB}} - \frac{\partial^2}{\partial Y^{(A}\partial U^{B)}} \right\} T(U,\, Y|X) = 0$$

$T(U,\, Y|X)$: generalized stress tensor. Rank-two equation is obeyed by

$$T(U,\, Y\,|X) = \sum_{i=1}^{N} C_{+\,i}(Y-U|X)\, C_{-\,i}(U+Y|X)$$

Rank-two fields: bilocal fields in the twistor space.

Dynamical currents (primaries) are Gelfond, MV (2003)

$$J(U|X) = T(U,0\,|X)\,, \qquad \tilde{J}(Y|X) = T(0,Y|X)$$

$$J^{asym}(U,Y|X) = (U^A Y^B - U^B Y^A)\Big(\frac{\partial^2}{\partial U^A \partial Y^B} T(U,Y|X)\Big|_{U^A=Y^A=0} \Big)$$

In the $3d$ case of $M = 2$ $A, B \to \alpha, \beta$. $J(U|X)$ generates $3d$ currents of all integer and half-integer spins

$$J(U|X) = \sum_{2s=0}^{\infty} U^{\alpha_1} \dots U^{\alpha_{2s}} J_{\alpha_1 \dots \alpha_{2s}}(X)\,, \qquad \tilde{J}(U|X) = \sum_{2s=0}^{\infty} U^{\alpha_1} \dots U^{\alpha_{2s}} \tilde{J}_{\alpha_1 \dots \alpha_{2s}}(X)$$

$$J^{asym}(U, Y|X) = U_\alpha Y^\alpha J^{asym}(X)$$

$$\Delta J_{\alpha_1 \dots \alpha_{2s}}(X) = \Delta \tilde{J}_{\alpha_1 \dots \alpha_{2s}}(X) = s + 1 \qquad \Delta(J^{asym}(X)) = 2$$

Differential equations: conventional conservation condition

$$\frac{\partial}{\partial X^{\alpha\beta}} \frac{\partial^2}{\partial U_\alpha \partial U_\beta} J(U|X) = 0\,, \qquad \frac{\partial}{\partial X^{\alpha\beta}} \frac{\partial^2}{\partial Y_\alpha \partial Y_\beta} \tilde{J}(Y|X) = 0$$

To define conserved charges, Fourier transform $T(U, Y|X)$

$$\widetilde{T}(\mathcal{W}, Y|X) \;=\; (2\pi)^{-M/2} \int_{\mathbb{R}^M} d^M U \;\; \exp\left(-i\;\mathcal{W}_C U^C\right) T(U, Y|X)$$

$$\left(\frac{\partial}{\partial X^{AB}} + i\mathcal{W}_{(A} \frac{\partial}{\partial Y^{B)}} \right) \widetilde{T} = 0$$

$2M$–form

$$\Omega^{2M}(T) = \Big(d\mathcal{W}_A \wedge \Big(i\,\mathcal{W}_B d\,X^{AB} - d\,Y^A\Big)\Big)^M \ \tilde{T}(\mathcal{W},\,Y\,|X)$$

is closed in $\mathcal{M}_M \times \mathbb{R}^M(\mathcal{W}_B) \times \mathbb{C}^M(Y^A)$

The charge

$$q = q(T) = \int_{\Sigma^{2M}} \Omega^{2M}(T)$$

is independent of local variations of a $2M$-dimensional surface Σ^{2M}.

Remarkable output: conserved charges can be expressed as integrals over the twistor space $\mathbb{T}$

Solutions of current equation form a commutative algebra

$$\eta(\mathcal{W},\,Y\,|X) = \varepsilon(\mathcal{W}_A,\,Y^C - i\,X^{CB}\,\mathcal{W}_B)\,, \qquad \tilde{T}_\eta(\mathcal{W},\,Y\,|X) = \eta(\mathcal{W},\,Y\,|X)\tilde{T}(\mathcal{W},\,Y\,|X$$

$\eta(\mathcal{W},\,Y\,|X)$ is a polynomial parameter representing global HS symmetry.

$q(\tilde{T}_\eta)$ with various $\eta(\mathcal{W},\,Y\,|X)$ generate complete set of conformal HS conserved charges. $M=2$: all conserved charges built from bilinears of free $3d$ massless fields.

$3d$ Conformal setup

For manifest conformal invariance introduce new oscillators

$$y^+_\alpha = \frac{1}{2}(y_\alpha - i\bar{y}_\alpha)\,, \qquad y^-_\alpha = \frac{1}{2}(\bar{y}_\alpha - iy_\alpha)\,, \qquad [y^-_\alpha\,, y^{+\beta}]_* = \delta^\beta_\alpha$$

$3d$ conformal realization of the algebra $sp(4;\mathbb{R}) \sim o(3,2)$

$$L^\alpha{}_\beta = y^{+\alpha} y^-_\beta - \frac{1}{2}\delta^\alpha_\beta y^{+\gamma} y^-_\gamma\,, \qquad D = \frac{1}{2} y^{+\alpha} y^-_\alpha$$

$$P_{\alpha\beta} = i y^-_\alpha y^-_\beta\,, \qquad K^{\alpha\beta} = -i y^{+\alpha} y^{+\beta}$$

Conformal weight of the HS gauge fields:

$$[D, \omega(y^\pm|X)] = \frac{1}{2}\left(y^{+\alpha}\frac{\partial}{\partial y^{+\alpha}} - y^-_\alpha \frac{\partial}{\partial y^-_\alpha}\right)\omega(y^\pm|X)\,.$$

Pullback $\widehat{\omega}(y^\pm|x)$ of $\omega(y^\pm|x)$ to Σ gives a set of $3d$ conformal HS gauge fields.

Conformal frame

D in the twisted adjoint representation is realized by the second-order operator

$$\{D\,,C\}_* = \left(y^{+\alpha}y^-_\alpha - \frac{1}{4}\frac{\partial^2}{\partial y^{+\alpha}\partial y^-_\alpha}\right) C$$

Fields C inherited from AdS_4 theory are not manifestly conformal.

Conformal frame: Wick star product

$$(f_N \star g_N)(y^\pm) = \int \mu(u^\pm) \exp(-u^-_\alpha u^{+\alpha}) f_N(y^+, y^- + u^-) g_N(y^+ + u^+, y^-)$$

$$f_N(y^\pm) = \exp -\frac{1}{2}\epsilon^{\alpha\beta}\frac{\partial^2}{\partial y^{-\alpha}\partial y^{+\beta}} f(y^\pm)$$

$$\{D_N\,,\ldots\}_\star = \frac{1}{2}\left(y^{+\alpha}\frac{\partial}{\partial y^{+\alpha}} + y^{-\alpha}\frac{\partial}{\partial y^{-\alpha}}\right) + y^-_\alpha y^{+\alpha} + 1$$

$$T(y^\pm|x) = \exp -y^-_\alpha y^{+\alpha} C_N(y^\pm|x)$$

$$\star \quad D_N(T(y^\pm)) = \frac{1}{2}\left(y^{+\alpha}\frac{\partial}{\partial y^{+\alpha}} + y^{-\alpha}\frac{\partial}{\partial y^{-\alpha}} + 2\right) T(y^\pm)$$

Holographic locality at infinity

AdS_4 **foliation:** $x^n = (\mathbf{x}^a, z)$ **where** $\mathbf{x}^a$ **are coordinates of leafs** $(a = 0, 1, 2)$ **while** z **is a foliation parameter.**

Poincaré coordinates

$$W = \frac{i}{z} d\mathbf{x}^{\alpha\beta} y^-_\alpha y^-_\beta - \frac{dz}{2z} y^-_\alpha y^{+\alpha}$$

$$e^{\alpha\dot\alpha} = \frac{1}{2z} dx^{\alpha\dot\alpha}\,, \qquad \omega^{\alpha\beta} = -\frac{i}{4z} d\mathbf{x}^{\alpha\beta}\,, \qquad \bar\omega^{\dot\alpha\dot\beta} = \frac{i}{4z} d\mathbf{x}^{\dot\alpha\dot\beta}$$

$$\left[d_{\mathbf{x}} + \frac{i}{z} d\mathbf{x}^{\alpha\beta} \left(y_\alpha \frac{\partial}{\partial y^\beta} - \bar y_\alpha \frac{\partial}{\partial \bar y^\beta} + y_\alpha \bar y_\beta - \frac{\partial^2}{\partial y^\alpha \partial \bar y^\beta} \right) \right] C(y, \bar y | \mathbf{x}, z) = 0$$

Rescaling y^α and $\bar y^{\dot\alpha}$ **via**

$$C(y, \bar y | \mathbf{x}, z) = z \exp(y_\alpha \bar y^\alpha) T(w, \bar w | \mathbf{x}, z)\,,$$

$$w^\alpha = z^{1/2} y^\alpha\,, \qquad \bar w^\alpha = z^{1/2} \bar y^\alpha$$

$T(w, \bar w | \mathbf{x}, z)$ **satisfies the** $3d$ **conformal invariant current equation**

$$\left[d_{\mathbf{x}} - i d\mathbf{x}^{\alpha\beta} \frac{\partial^2}{\partial w^\alpha \partial \bar w^\beta}\right] T(w, \bar w | \mathbf{x}, z) = 0$$

Connections

Setting

$$W^{jj}(y^{\pm}|\mathbf{x},z) = \Omega^{jj}(v^{-},w^{+}|\mathbf{x},z)$$

$$v^{\pm} = z^{-1/2}y^{\pm}\,, \qquad w^{\pm} = z^{1/2}y^{\pm}$$

explicit z–dependence disappears

$$D_{\mathbf{x}}\Omega^{jj}(v^{-},w^{+}|\mathbf{x},z) = \left(d_{\mathbf{x}} + 2id\mathbf{x}^{\alpha\beta}v^{-}_{\alpha}\frac{\partial}{\partial w^{+\beta}}\right)\Omega^{jj}(v^{-},w^{+}|\mathbf{x},z)$$

Using

$$w_{\alpha} = w^{+}_{\alpha} + izv^{-}_{\alpha}\,, \qquad \bar{w}_{\alpha} = iw^{+}_{\alpha} + zv^{-}_{\alpha}$$

free HS equations take the form

$$D_{\mathbf{x}}\Omega^{jj}_{\mathbf{x}}(v^{-},w^{+}|\mathbf{x},z) =$$
$$d\mathbf{x}_{\alpha}{}^{\gamma}d\mathbf{x}_{\beta\gamma}\frac{\partial^2}{\partial w^{+}_{\alpha}\partial w^{+}_{\beta}}\left(\eta T^{j\,1-j}(w^{+}+izv^{-},0\mid \mathbf{x},z) - \bar{\eta}T^{1-j\,j}(0,iw^{+}+zv^{-}\mid \mathbf{x},z)\right)$$

$z \to 0$ limit

Setting

$$\mathcal{T}^{jj}(w^+, w^-|\mathbf{x}, 0) = \eta T^{j\,1-j}(w^+, w^- \mid \mathbf{x}, 0) - \bar{\eta} T^{1-j\,j}(-iw^-, iw^+ \mid \mathbf{x}, 0)$$

$\star$ $$D_{\mathbf{x}} \Omega^{jj}_{\mathbf{x}}(v^-, w^+|\mathbf{x}, 0) = d\mathbf{x}_\alpha{}^\gamma d\mathbf{x}_{\beta\gamma} \frac{\partial^2}{\partial w^{+\alpha} \partial w^{+\beta}} \mathcal{T}^{jj}(w^+, 0 \mid \mathbf{x}, 0)\,,$$

$\star$ $$\left[d_{\mathbf{x}} - i d\mathbf{x}^{\alpha\beta} \frac{\partial^2}{\partial w^{+\alpha} \partial w^{-\beta}} \right] T^{j\,1-j}(w^+, w^-|\mathbf{x}, 0) = 0\,.$$

Towards nonlinear $3d$ conformal HS theory

Conformal HS theory is nonlinear since conformal HS curvatures inherited from the AdS_4 HS theory are non-Abelian   Fradkin, Linetsky (1990)

$$R_{\mathbf{x}\mathbf{x}}(v^-, w^+ \mid \mathbf{x}) = d_{\mathbf{x}}\Omega_{\mathbf{x}}(v^-, w^+ \mid \mathbf{x}) + \Omega_{\mathbf{x}}(v^-, w^+ \mid \mathbf{x}) \star \Omega_{\mathbf{x}}(v^-, w^+ \mid \mathbf{x})$$

It is important

$$[v^-_\alpha , w^{+\beta}]_\star = \delta^\beta_\alpha$$

The equation on 0-forms deforms to nonlinear twisted adjoint representation

$$dT(w^\pm|x) + \Omega(\frac{\partial}{\partial w^{+\beta}}, w^+_\alpha) \circ T(w^\pm|x) - T(w^\pm|x) \circ \Omega(-i\eta\frac{\partial}{\partial w^{-\alpha}}, -i\eta w^-|x) = O(T^2)\,.$$

Matter fields can be added via the Fock module

$$(d + \Omega_0(v^-, w^+|\mathbf{x})) \star C^i(w^+|\mathbf{x}) \star F = 0\,.$$

Free CFT_3 reduction

The unfolded equation

$$D_{\mathbf{x}}\Omega^{jj}_{\mathbf{x}}(v^-, w^+|\mathbf{x}, 0) = \mathcal{H}^{\alpha\beta}\frac{\partial^2}{\partial w^{+\alpha}\partial w^{+\beta}}\mathcal{T}^{jj}(w^+, 0 \mid \mathbf{x}, 0)$$

remains free if

$$\mathcal{T}^{jj} = 0 \qquad \longrightarrow J^{asym} = 0 \qquad \text{or} \qquad J^{sym} = 0$$

depending on whether A-model or B-model is considered. For these cases the model remains free in accordance with the Klebanov-Polyakov, Sezgin-Sundell conjecture.

Free models are equivalent to the reductions of the HS theory with respect to involution $y \leftrightarrow \bar{y}$ which is possible for the A and B models.

For HS theory with general phase η parameter such reduction is not possible: no realization as a free conformal theory.

Non-Abelian contribution of superconformal HS connections has to be taken into account.

Higher-spin theory and quantum mechanics

rank-one equation in $\mathcal{M}_M$ can be rewritten in the form

$$\Big(ih\frac{\partial}{\partial X^{AB}} + \frac{h^2}{2m}\frac{\partial^2}{\partial Y^A \partial Y^B}\Big)\Psi(Y|X) = 0$$

Algebra of symmetries: algebra of polynomials of $P_A = \frac{\partial}{\partial Y^A}$ and Y^B:

conformal HS algebra. $sp(2M)$:

$$K^{AB} = Y^A Y^B\,, \qquad L^A{}_B = \{Y^A\,, P_B\}\,, \qquad P_{AB} = P_A P_B$$

Time-like directions in $\mathcal{M}_M$ are associated with positive-definite X^{AB}

$$X^{AB} = tM\delta^{AB}$$

Restriction to t gives M-dimensional Schrodinger equation

$$\Big(ih\frac{\partial}{\partial t} + \frac{h^2}{2m}\delta^{AB}\frac{\partial^2}{\partial Y^A \partial Y^B}\Big)\Psi(Y|t) = 0$$

Y^A are now interpreted as Galilean coordinates.

In unfolded dynamics it is easy to introduce coordinates in which any symmetry h of a given system acts geometrically by introducing a non-zero flat connection of h. Different symmetries require different spaces and connections. Description of the same system in different space-times gives holographically dual theories.

Being obvious in unfolded dynamics, where it refers to the same twistor space (Y^A) in other approaches holographic duality may look obscure.

Maximal finite dimensional symmetry algebra $sph(M|\mathbb{R})$ Valenzuela (2009)

$$T_{AB} = -\frac{i}{2} Y_A Y_B \,, \qquad t_A = Y_A$$

$$[T_{AB}\,, T_{CD}] = C_{BC} T_{AD} + C_{AC} T_{BD} + C_{BD} T_{AC} + C_{AD} T_{BC}$$

$$[T_{AB}\,, t_C] = C_{BC} t_A + C_{AC} t_B \,, \qquad [t_A\,, t_B] = 2i C_{AB}$$

Relativistic and nonrelativistic symmetries of Schrodinger equation belong to $sph(M|\mathbb{R})$. Each symmetry acts geometrically in respective space.

What if the system is deformed by a potential? Formally, this does not affect the consideration much. In presence of potential $U(Y)$ the equation

$$\Big(ih\frac{\partial}{\partial t}+\frac{h^2}{2m}\delta^{AB}\frac{\partial^2}{\partial Y^A\partial Y^B}-U(Y)\Big)\Psi(Y|t)=0$$

remains linear, hence exhibiting infinite symmetries. It can be interpreted as flatness condition

$$D\Psi(Y|t)=0\,,\qquad D=dt\frac{\partial}{\partial t}+\Omega\,,\qquad \Omega=ih^{-1}dtH\,,\qquad H=-\frac{h^2}{2m}\delta^{AB}\frac{\partial^2}{\partial Y^A\partial Y^B}$$

In the $1d$ case with the single coordinate t, any connection is flat. Hence it can be represented in the pure gauge form which is simply

$$\Omega=\exp -ih^{-1}Ht\,d\,\exp ih^{-1}Ht$$

Any HS geometry is holographically dual to some quantum mechanics.

For example, AdS geometry is dual to harmonic potential

$$U(Y) = \frac{1}{2} m\omega^2 Y^A Y^B \delta_{AB}$$

where $-\Lambda \sim \lambda^2$

$$\frac{1}{2} m\omega^2 = \lambda^2 .$$

dS geometry is holographically dual to the inverted harmonic potential

not too surprisingly in the context of inflation.

Conclusions

Holographic duality relates theories that have equivalent unfolded formulation: equivalent twistor space description.

Beyond $1/N$

AdS_4 HS theory is dual to nonlinear $3d$ conformal HS theory of $3d$ currents

Maldacena-Zhiboedov theorem is escaped by virtue of boundary gauge conformal HS symmetries

Both of holographically dual theories are HS theories of gravity

Relativistic HS field equations are holographically dual to nonrelativistic quantum mechanics.

Holography at any surface is nonlocal

To do

Nonlinear $3d$ conformal HS theory

Actions

Correlators

AdS_3/CFT_2 **and Gaberdiel-Gopakumar conjecture**

GGI Program

“Higher Spins, Strings and Dualities”

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Florence, March 18 - May 10, 2013

Organizers:

D.Francia, M.Gaberdiel, I.Klebanov, A.Sagnotti, D.Sorokin, M.Vasiliev