UNITARITY IN HIGHER DERIVATIVE THEORIES

Ghosts of Critical Gravity, M.P., M. M. Roberts, arXiv:1104.0674 [hep-th]. Phys.Rev. D84 (2011) 024013.

Unitary Truncations and Critical Gravity: a Toy Model, E.A. Bergshoeff, S. de Haan, W. Merbis, M.P, J. Rosseel, arXiv:1201.0449 [hep-th]

CAN HIGHER DERIVATIVE THEORIES OF GRAVITY BE UNITARY, OR AT LEAST NON-UNITARY IN AN "INTERESTING" WAY?

SCALAR PRODUCT IN FIELD THEORY

DEFINED BY THE LAGRANGIAN:

$$\int d^n x[\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2] \quad \Rightarrow \quad \langle \psi | \chi \rangle = \frac{1}{2} \int d^{n-1} x (\psi^+)^* \overleftrightarrow{\partial_t} \chi^+$$

POSITIVE FREQUENCY PART OF ON-SHELL ϕ

GENERALIZES TO

$$\frac{1}{2} \int d^n x A_{ij} [\partial_\mu \phi^i \partial^\mu \phi^j + B_{ij} \phi^i \phi^j]$$

$$\Rightarrow \quad \langle \psi | \chi \rangle = \frac{1}{2} \int d^{n-1} x A_{ij} (\psi^{i+})^* \overleftrightarrow{\partial_t} \chi^{j+}$$

ENOUGH FOR FINDING PERTURBATIVE UNITARITY OF HIGHER DERIVATIVE SCALAR THEORIES

CHOOSE:

$$A_{ij} = \begin{pmatrix} 0 & . & . & 0 & 1 \\ 0 & . & . & 1 & 0 \\ . & . & . & . & . \\ 0 & 1 & . & . & . \\ 1 & 0 & . & . & 0 \end{pmatrix} \qquad B_{ij} = \begin{pmatrix} 0 & . & . & 1 & m^2 \\ 0 & . & 1 & m^2 & 0 \\ . & . & . & . & . \\ 1 & m^2 & . & . & 0 \\ m^2 & 0 & . & . & 0 \end{pmatrix}$$

E.O.M:

$$(\Box - m^2)\phi^i = \phi^{i-1}$$

$$(\Box - m^2)\phi^1 = 0$$

$$\Longrightarrow \quad (\Box - m^2)^N \phi^N = 0$$

WHEN DO WE HAVE UNITARITY?

1) NEVER

2) ALWAYS

3) SOMETIMES

WHEN DO WE HAVE UNITARITY?

1) NEVER A_{ij} HAS NEGATIVE EIGENVALUES

2) ALWAYS

3) SOMETIMES

WHEN DO WE HAVE UNITARITY?

1) NEVER A_{ij} HAS NEGATIVE EIGENVALUES

2) ALWAYS

A_{ij} HAS POSITIVE SUBSPACE: $\phi^i + \phi^{N+1-i}$

3) SOMETIMES

WHEN DO WE HAVE UNITARITY?

1) NEVER A_{ij} HAS NEGATIVE EIGENVALUES

2) ALWAYS

A_{ij} HAS POSITIVE SUBSPACE: $\phi^i + \phi^{N+1-i}$

3) SOMETIMES: UNDER TIME EVOLUTION $\phi^i \to \phi^i + \sum_{j<i} \phi^j$

SO SUBSPACE IS NOT CLOSED UNDER TIME EVOLUTION EXCEPT WHEN...

N +1 -i = i $\Rightarrow$ i=(N+1)/2

PRECISELY: $\phi^i = 0, \qquad i > (N+1)/2$

DEFINES A NON-NEGATIVE SUBSPACE WITH METRIC PROPORTIONAL TO

$$A_{ij}^{reduced} = \begin{pmatrix} 1 & . & . & 0 \\ 0 & . & . & 0 \\ . & . & . & . \\ 0 & 0 & . & 0 \end{pmatrix}$$

MODDING OUT BY ZERO MODES $\phi^j, \qquad j < (N+1)/2$

WE DEFINE A POSITIVE DEFINITE HILBERT SPACE

APPLICATION TO ADS/CFT

ANTI DE SITTER METRIC (POINCARE):

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu)$$

E.O.M. FOR OUR SCALAR SYSTEM:

$$(\Box - m^2)\phi^1 = 0 \quad \Rightarrow \quad \phi^1 \sim z^{\Delta_-}\varphi(x) \qquad z \to 0$$

$$\Delta_\pm = 3/2 \pm \sqrt{(Lm)^2 + 9/4}$$

$$\phi^i \sim z^{\Delta_-} \log^{i-1}(z/\lambda)\varphi(x) \qquad z \to 0$$

THE 3D BOUNDARY FIELDS $\varphi^i(x)$ ARE SOURCES FOR OPERATORS IN DUAL CFT

$$S[\varphi^i(x)] = W[\varphi^i(x)]$$

$$\delta^n W[\varphi(x)] = \int d^3x^1 ... \int d^3x^n \delta\varphi(x^1)...\delta\varphi(x^n)\langle O(x^1)...O(x^n)\rangle$$

CORRELATORS OF TWO OPERATORS IN DUAL CTF CAN BE COMPUTED FROM THE FREE ACTION

BULK FIELD DETERMINED BY BOUNDARY VALUE USING BOUNDARY-TO-BULK GREEN'S FUNCTION

$$\phi^i(x,z) = \sum_{j=1}^{i} \int d^3y \varphi^j(x) G^{i+1-j}(z,x,y)$$

$$(\Box - m^2)G^i = G^{i-1}, \qquad G^0 \equiv 0$$

SUBSTITUTE IN ON-SHELL ACTION, REGULARIZE AND COMPUTE:

$$\frac{\delta S}{\delta \varphi^i(x) \delta \varphi^j(y)} = \langle O^{N-i}(x) O^{N-j}(y) \rangle$$

WE OBTAIN THE TWO-POINT FUNCTIONS OF A LOGARITHMIC CFT

$$\langle O^i(x) O^j(y) \rangle = \text{constant}\ \frac{\log^{i+j-N-1}|x-y|}{|x-y|^{2\Delta_+}}$$

THE TWO-POINT FUNCTION VANISHES WHEN i+j-N-1 <0,
SO THE SUBSPACE SPANNED BY

$$O^i(x), \qquad i \leq (N+1)/2$$

HAS ONE NON-LOGARITHMIC 2-POINT FUNCTION AND MANY NULL OPERATORS

THAT IS THE STRUCTURE OF A STANDARD CTF WHEN WE MOD OUT THE NULL OPERATORS

FOR N ODD LOGARITHMIC CFTS EXHIBIT THE SAME STRUCTURE OF GAUGE THEORIES:

- A SUBSPACE OF A LARGER INDEFINITE-METRIC SPACE CONTAINS ONLY PHYSICAL AND NULL STATES
- NULL STATES CAN BE REMOVED BY DEFINING PHYSICAL STATES AS EQUIVALENT IF THEY DIFFER BY NULL STATES:

$$|O^{(N+1)/2}\rangle \sim |O^{(N+1)/2}\rangle + \sum_{j<(N+1)/2} c_j |O^j\rangle$$

THIS IS NOT POSSIBLE WHEN N IS EVEN

N EITHER EVEN OR ODD CARRY MULTI-LOGARITHMIC REPRESENTATIONS OF THE CONFORMAL GROUP

$$SO(d-1,2) \supset SO(d-1) \times SO(2)$$

THIS IS GLOBAL TIME TRANSLATIONS WITH GENERATOR H

H ACTS ON SCALAR FIELDS IN ADS AS:

$$H\phi^i = \Delta_- \phi^i + \sum_{j<i} \phi^j$$

THEY CARRY A (MULTI)-LOGARITHMIC REPRESENTATION OF SO(d-1,2)

LOGARITHMIC REPS OF THE CONFORMAL GROUP AND LOGARITHMIC CFTS ARE INTERESTING THEORIES IN STATISTICAL MECHANICS (CONDENSED MATTER?)

CFTS HAVE AN ENERGY MOMENTUM TENSOR. IN ADS THIS MEANS THAT THE METRIC IS DYNAMICAL.

WHAT IF THE EQUATIONS GOVERNING THIS METRIC ARE THEMSELVES HIGHER DERIVATIVE?

COULD ONE FIND A HIGHER DERIVATIVE UNITARY THEORY OF GRAVITY?

AN INTERESTING FAILURE: CRITICAL GRAVITY

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g}[R - 2\Lambda + aG_{\mu\nu}G^{\mu\nu} + bR^2]$$

TUNE COEFFICIENTS a, b TO HAVE ONLY MASSLESS SPIN 2 MODES PROPAGATING

WRITE AS:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g}[R + 6m^2 - (1/2)f^{\mu\nu}G_{\mu\nu} + (m^2/3)(f_{\mu\nu}f^{\mu\nu} - f^2)]$$

LINEARIZE AROUND ADS BACKGROUND. USE BIANCHI IDENTITIES TO WRITE QUADRATIC ACTION AS:

$$S = \frac{1}{6m^2} \int d^4x \sqrt{-g}[D_\lambda h^{\mu\nu} D^\lambda k_{\mu\nu} - 2m^2 h^{\mu\nu} k_{\mu\nu} - (1/3)k_{\mu\nu}k^{\mu\nu}]$$

THE EQUATIONS OF MOTION ARE

$$(D_\lambda D^\lambda + 2m^2)h_{\mu\nu} = -(2/3)k_{\mu\nu}, \qquad (D_\lambda D^\lambda + 2m^2)k_{\mu\nu} = 0$$

$$h_{\mu\nu}, k_{\mu\nu} \text{ transverse-traceless}$$

LAGRANGIAN AND EQUATIONS OF MOTION HAVE THE SAME STRUCTURE AS THE SCALAR SYSTEM WE STUDIED, WITH N=2

SO THE SCALAR PRODUCT IS

$$\langle\psi|\psi'\rangle = \frac{1}{2}\int d^3x\sqrt{-g}g^{00}[(k^+)^* \overleftrightarrow{D}_t h'^+ + (h^+)^* \overleftrightarrow{D}_t k'^+]$$

IT CANNOT HAVE A POSITIVE INVARIANT SUBSPACE BECAUSE k SOLVES EINSTEIN EQUATIONS AND THE SCALAR PRODUCT OF EINSTEIN MODES DOES NOT VANISH IN DIMENSION d LARGER THAN 3.

EXCEPTION: RESTRICT THE LOGARITHMIC MODES

SOLUTIONS OF 3D EINSTEIN EQUATIONS THAT ARE PURE GAUGE:

$$k_{\mu\nu} = D_\mu A_\nu + \mu \leftrightarrow \nu$$

SCALAR PRODUCT OF A (k,h) MODE WITH A (k',h'):

$$\langle \psi | \psi' \rangle = \frac{1}{2T} \int d^D x \sqrt{-g} \xi^\mu [(k^+)^* \overleftrightarrow{D}_\mu h'^+ + (h^+)^* \overleftrightarrow{D}_\mu k'^+], \qquad \xi_\mu = (1,0,0,0)$$

NOT THE TIMELIKE KILLING VECTOR, BUT OBEYS $\quad D_\mu \xi^\mu = 0, \qquad D_\mu \xi_\nu = D_\nu \xi_\mu$

INTEGRATIONS BY PART PLUS EQUATIONS OF MOTION

$$\left(D_\lambda D^\lambda + \frac{2\Lambda}{(d-2)}\right) A_\mu = 0, \qquad \left(D_\lambda D^\lambda - \frac{4\Lambda}{(d-2)(d-1)}\right) h_{\mu\nu} = k_{\mu\nu}$$

$$\langle\psi|\psi'\rangle \propto \int d^D x \sqrt{-g}\,\xi^\mu[(k^+_{\mu\nu})^* A'^{\nu+} - (A^{\nu+})^* k'^+_{\mu\nu}] =$$

$$= \int d^D x \sqrt{-g}\,\xi^\mu[(D_{(\mu} A^+_{\nu)})^* A'^{\nu+} - (A^{\nu+})^* D_{(\mu} A'^+_{\nu)}] =$$

$$= \int d^D x \sqrt{-g}\,\xi^\mu[(D_\mu A^+_\nu)^* A'^{\nu+} - (A^{\nu+})^* D_\mu A'^+_\nu]$$

THIS IS THE CANONICAL NORM FOR A SPIN ONE FIELD:

THEORY PROPAGATES ONLY A SPIN ONE IN THE BULK.

AT NONLINEAR LEVEL THEORY IS SIMPLY INDUCED GRAVITY .... WITH ITS USUAL PROBLEMS!

$$f_{\mu\nu} = D_{(\mu}A_{\nu)} + c g_{\mu\nu}$$

$$S \propto \int d^D x \sqrt{-g} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\Lambda}{2(d-2)} A_\mu A^\mu \right]$$

NO EINSTEIN TERM:

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0 \quad \Longrightarrow \quad T_{\mu\nu} = 0$$

N=3 UNITARY HIGHER-DERIVATIVE GRAVITY?

YES, BUT SO FAR ONLY AT QUADRATIC LEVEL

AN ACTION THAT WORKS IS:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[R - 2\Lambda + \frac{1}{2|\Lambda|} C^{\mu\nu\rho\sigma} \left(1 + \frac{1}{4|\Lambda|} \Box \right) C^{\mu\nu\rho\sigma} \right]$$

Polycritical Gravities, Teake Nutma, arXiv:1203.5338 [hep-th]

L. Apolo, M. Porrati (unpublished)

EXTENDING THIS RESULT TO NON-LINEAR ORDER NEEDS SEVERAL HIGHLY NON-TRIVIAL CHECKS

THE PHYSICAL INVARIANT SUBSPACE MUST BE LEFT INVARIANT BY THE FULL NON-LINEAR EVOLUTION. WE NEED A POSITIVE DEFINITE CONSERVED CHARGE TO ENSURE THIS, BUT THE ONLY NATURAL ONE WE HAVE IS NOT POSITIVE, NOT EVEN AT QUADRATIC ORDER

$$E = \frac{i}{2}\int d^3x\sqrt{-g}g^{00}[(\dot{k}^+)^* \overleftrightarrow{D}_t\, h^+ + (\dot{h}^+)^* \overleftrightarrow{D}_t\, k^+]$$

h IS THE METRIC PERTURBATION,

k SOLVES THE HOMOGENEOUS EINSTEIN EQ.

EXTENDING THIS RESULT TO NON-LINEAR ORDER NEEDS SEVERAL HIGHLY NON-TRIVIAL CHECKS

THE PHYSICAL INVARIANT SUBSPACE MUST BE LEFT INVARIANT BY THE FULL NON-LINEAR EVOLUTION. WE NEED A POSITIVE DEFINITE CONSERVED CHARGE TO ENSURE THIS, BUT THE ONLY NATURAL ONE WE HAVE IS NOT POSITIVE, NOT EVEN AT QUADRATIC ORDER

$$E = \frac{i}{2}\int d^3x\sqrt{-g}g^{00}[(\dot{k}^+)^* \overleftrightarrow{D}_t h^+ + (\dot{h}^+)^* \overleftrightarrow{D}_t k^+]$$

h IS THE METRIC PERTURBATION,

k SOLVES THE HOMOGENEOUS EINSTEIN EQ.

THE NULL MODES (h THAT SOLVE THE HOMOGENEOUS EINSTEIN EQUATIONS) MUST BE GAUGE MODES AT NON-LINEAR ORDER

SO WE LEAVE THIS DIFFICULT PROBLEM AND COME BACK TO SCALARS!

IN 4D THERE IS A SPECIAL VALUE OF THE SCALAR'S MASS THAT MAKES A SUB-SECTOR OF THE N=2 THEORY UNITARY

$$m^2 = \frac{5}{4L^2}, \qquad \Delta_+ = 5/2, \qquad \Delta_- = 1/2$$

$$S = \int d^4x \sqrt{-g} \left[g^{\mu\nu} \partial_\mu \psi \partial_\nu \phi - \frac{1}{2}\psi^2 - \frac{5}{4}\psi\phi \right] + \lim_{z\to 0} \int d^3x \frac{1}{z^3} g^{ab} \left[\partial_a \phi \partial_b \phi - \psi\phi \right]$$

THIS IS THE FLATO FRONSDAL SINGLETON ACTION (L=1)

WITH BOUNDARY TERM, EULER LAGRANGE EQUATIONS GIVE:

$$\phi \sim z^{5/2}, \qquad \phi \sim z^{5/2} \log z, \qquad \psi \sim z^{5/2}$$

GAUGE

UNPHYSICAL

$$\phi \sim z^{1/2} \varphi(x), \qquad \Box_3 \varphi(x) = 0$$

PHYSICAL MODE: FREE SCALAR IN 3D: SINGLETON

FF IS A BULK ACTION FOR THE BOUNDARY SINGLETON DOF

THE SCALAR PRODUCT IS

$$\langle \chi | \chi' \rangle = \frac{1}{2} \int d^3x \sqrt{-g} g^{00} [(\phi^+)^* \overleftrightarrow{D}_t \psi'^+ + (\psi^+)^* \overleftrightarrow{D}_t \phi'^+] +$$

$$\lim_{z \to 0} \int d^2x (\varphi^+)^* \overleftrightarrow{\partial}_t \varphi'^+$$

THIS VANISHES ON THE PHYSICAL SUBSPACE

ON THE PHYSICAL SUBSPACE THIS IS A POSITIVE SCALAR PRODUCT

THE SCALAR PRODUCT IS

$$\langle\chi|\chi'\rangle = \frac{1}{2}\int d^3x\sqrt{-g}g^{00}[(\phi^+)^* \overleftrightarrow{D}_t \psi'^+ + (\psi^+)^* \overleftrightarrow{D}_t \phi'^+]+$$

$$\lim_{z\to 0}\int d^2x(\varphi^+)^* \overleftrightarrow{\partial}_t \varphi'^+$$

THIS VANISHES ON THE PHYSICAL SUBSPACE

ON THE PHYSICAL SUBSPACE THIS IS A POSITIVE SCALAR PRODUCT

BRS:

$$S \to S + \int d^4x\sqrt{-g}[g^{\mu\nu}\partial_\mu b\partial_\nu c - (5/4)bc]$$

$$\delta_B\phi = c, \qquad \delta b = \psi$$

BULK COMPOSITE FIELDS FOR VASILIEV THEORY?

$$J_{\mu_1\ldots\mu_n} = \delta(z)\varphi\partial_{(\mu_1}\ldots\partial_{\mu_n)}\varphi + Q_B[b\partial_{(\mu_1}\ldots\partial_{\mu_n)}\phi]$$

BOUNDARY TERM, CLOSED

BULK TERM, EXACT

BULK ACTION OUT OF $\quad O = \text{ bilinear in } \phi, \psi$?

$$S = \langle 0|O Q_B O|0\rangle + \langle 0|O * O * O|0\rangle + \ldots.$$

RANDALL & SUNDRUM MEET VASILIEV

ADS SPACE CUTOFF AT $z > \epsilon$

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \eta_{mn} dx^m dx^n)$$

MASSLESS MODES ON 4D BOUNDARY:

$$\psi_{m_1,..m_s}(z,x) = z^{4-\Delta-s}\hat{\psi}_{m_1,..m_s}(x)$$

KINETIC TERM FINITE = DYNAMICAL MASSLESS 4D FIELD

$$\int d^4x dz \sqrt{-g} g^{m_1 n_1} \ldots . g^{m_{s+1} n_{s+1}} \partial_{m_1} \psi_{m_2 \ldots m_{s+1}}(z,x) \partial_{n_1} \psi_{m_2 \ldots m_{s+1}}(z,x) + \ldots .$$

$$\propto \int_\epsilon^\infty dz z^{2s-3} z^{8-2s-2\Delta} \int d^4x \partial_{\mu_1} \hat{\psi}_{\mu_2 \ldots \mu_{s+1}}(x) \partial^{\mu_1} \hat{\psi}^{\mu_2 \ldots \mu_{s+1}}(x) + \ldots ..$$

VIOLATES MANY NO GO THEOREMS!

LINEARIZED GAUGE TRANSFORMATION:

$$\delta\psi_{m_1,..m_s}(z,x) = D_{(m_1}\epsilon_{m_2...m_s)_T}(z,x) \qquad \epsilon_{(m_1...m_{s-1})_T} = z^{2-2s}\hat{\epsilon}_{(m_1...m_{s-1})_T}$$

ACTION NOT INVARIANT UNDER THESE TRANSFORMATIONS

EXAMPLE: SPIN 3 COUPLING TO GRAVITON

$$\delta S_3 = -\frac{3}{\Lambda}\int d^4x \frac{L^5}{\epsilon^5} w_{\alpha\beta\gamma\delta} D^{(\mu}\epsilon^{\alpha\beta)_T} \overleftrightarrow{D}_z \psi_\mu^{\gamma\delta}.$$

WEYL TENSOR

THIS BOUNDARY TERM IS PRECISELY THE ANOMALY OF THE
4D HIGH-SPIN GAUGE TRANSFORMATION

GAUGE ANOMALY MEANS HIGH-SPIN FIELDS BECOME MASSIVE

BOUNDARY DESCRIPTION

MASSLESS FIELDS DUAL TO CONSERVED HIGH SPIN CURRENTS IN FREE O(N) MODELS

$$J_{\mu_1 \ldots . \mu_s} = \sum_{I=1}^{N} \phi^I \overleftrightarrow{\partial}_{(\mu_1} \ldots . . \overleftrightarrow{\partial}_{\mu_s)_T} \phi^I$$

THESE CURRENTS ARE NO LONGER CONSERVED IN THE PRESENCE OF 4D DYNAMICAL GRAVITY:

NON CONSERVED CURRENTS COUPLE TO MASSIVE FIELDS.

MASSES NATURALLY ARISE FROM BOUNDARY COUNTER TERMS NEEDED TO

CANCEL THE $1/\epsilon^2$ DIVERGENCES OF THE ON SHELL ACTION