Higher Spin Black Holes from CFT

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Overview of the talk:

- Higher spin black holes: [Kraus & Perlmutter '11]

$$\ln Z_{BH}(\hat{\tau},\alpha,\hat{\bar{\tau}},\bar{\alpha}) = \frac{i\pi c}{12\hat{\tau}}\Big[1-\frac{4}{3}\frac{\alpha^2}{\hat{\tau}^4}+\frac{400}{27}\frac{\lambda^2-7}{\lambda^2-4}\frac{\alpha^4}{\hat{\tau}^8} - \frac{1600}{27}\frac{5\lambda^4-85\lambda^2+377}{(\lambda^2-4)^2}\frac{\alpha^6}{\hat{\tau}^{12}} + \frac{32000}{81}\frac{20\lambda^6-600\lambda^4+6387\lambda^2-23357}{(\lambda^2-4)^3}\frac{\alpha^8}{\hat{\tau}^{16}}+\cdots\Big] + \text{rightmoving}$$

  where $\hat{\tau}$ is the modular parameter of the torus, α is the chemical potential of the spin-3 current, and λ indicates the bulk symmetry algebra: hs$[\lambda]$.

- High temperature:

$$\hat{\tau},\alpha\to 0 \qquad \text{and} \qquad \frac{\alpha}{\hat{\tau}^2} \quad \text{fixed}$$

The partition function

- From the CFT point of view, this amounts to calculate the partition function

$$Z_{CFT}(\hat{\tau}, \alpha, \hat{\bar{\tau}}, \bar{\alpha}) = \mathrm{Tr}\left(\hat{q}^{L_0 - \frac{c}{24}} y^{W_0} \hat{\bar{q}}^{\bar{L}_0 - \frac{c}{24}} \bar{y}^{\bar{W}_0}\right)$$

where $\hat{q} = e^{2\pi i \hat{\tau}}$ and $y = e^{2\pi i \alpha}$.

- The asymptotic symmetry algebra is $\mathcal{W}_\infty[\lambda]$.
- Checked already for $\lambda = 0, 1$. [Kraus & Perlmutter '11]
- Under the S-transformation: $\hat{\tau} \to -1/\tau$, $q = e^{2\pi i \tau} \to 0$, only the vacuum representation is needed in the leading order $\Longrightarrow$ splitting of the holomorphic/anti-holomorphic parts.
- Focusing on the holomorphic part, a general formula of the character under the S-transformation is unknown

$$\mathrm{Tr}_i\left(\hat{q}^{L_0 - \frac{c}{24}} y^{W_0}\right) \longrightarrow \sum_j S_{ij} \cdots \mathrm{Tr}_j\left(q^{L_0 - \frac{c}{24}} \cdots\right)$$

Our method:

- First we expand:

$$y^{W_0} = e^{2\pi i \alpha W_0} = 1 + (2\pi i)\alpha W_0 + \frac{(2\pi i)^2 \alpha^2}{2!} W_0^2 + \cdots$$

- The BTZ term: "background"

$$\mathrm{Tr}\left(\hat{q}^{L_0 - \frac{c}{24}}\right) = \sum_{ij} S_{ij} \mathrm{Tr}_j \left(q^{L_0 - \frac{c}{24}}\right) \sim \Big(\sum_i S_{i0}\Big) q^{-\frac{c}{24}}$$

$$\Longrightarrow \ln Z = -\frac{i\pi c}{12}\tau = \frac{i\pi c}{12\hat{\tau}} \qquad (\hat{\tau} \to 0)$$

- Second: apply S-transformation to each terms

$$\alpha \mathrm{Tr}\left(W_0 \hat{q}^{L_0 - \frac{c}{24}}\right) \to 0, \qquad \alpha^2 \,\mathrm{Tr}\left(W_0^2 \hat{q}^{L_0 - \frac{c}{24}}\right) \to \frac{c\alpha^2}{\hat{\tau}^5}$$

- Comparison to the gravity result

$$\ln Z_{BH}(\hat{\tau}, \alpha) = \frac{i\pi c}{12\hat{\tau}} \left[1 - \frac{4}{3}\frac{\alpha^2}{\hat{\tau}^4} + \frac{400}{27}\frac{\lambda^2 - 7}{\lambda^2 - 4}\frac{\alpha^4}{\hat{\tau}^8} + f(\lambda)\frac{\alpha^6}{\hat{\tau}^{12}} + \cdots\right]$$

$\mathcal{W}_\infty[\lambda]$ commutation relations

$$
\begin{aligned}
[W_m, W_n] =& 2(m-n)U_{m+n} + \frac{N_3}{12}(m-n)(2m^2+2n^2-mn-8)L_{m+n} \\
& + \frac{cN_3}{144}m(m^2-1)(m^2-4)\delta_{m+n,0} + \frac{8N_3}{c}(m-n)\Lambda^{(4)}_{m+n} \\
[W_m, U_n] =& (3m-2n)X_{m+n} + \frac{N_4}{15N_3}(n^3-5m^3-3mn^2+5m^2n-9n+17m)W_{m+n} \\
& - \frac{24N_4}{15cN_3}(7+17m-9n)\Lambda^{(5)}_{m+n} + \frac{84N_4}{15cN_3}\Theta^{(6)}_{m+n} \\
[W_m, X_n] =& (4m-2n)Y_{m+n} - \frac{N_5}{56N_4}(28m^3-21m^2n+9mn^2-2n^3-88m+32n)U_{m+n} \\
& + \frac{42N_5}{5cN_3^2}(2m-n)\Lambda^{(6)}_{m+n} + \cdots \\
[U_m, U_n] =& 3(m-n)Y_{m+n} + n_{44}(m-n)(-7+m^2-mn+n^2)U_{m+n} \\
& - \frac{N_4}{360}(m-n)(108-39m^2+3m^4+20mn-2m^3n-39n^2 \\
& + 4m^2n^2-2mn^3+3n^4)L_{m+n} - (m-n)\frac{N_4 n_q}{cN_3^2}\Lambda^{(6)}_{m+n} \\
& - \frac{cN_4}{4320}m(m^2-1)(m^2-4)(m^2-9)\delta_{m+n,0}
\end{aligned}
$$

Constants and the nonlinear terms

- The constants are

$$N_3 = \frac{16}{5}\sigma^2(\lambda^2 - 4)$$
$$N_4 = -\frac{384}{35}\sigma^4(\lambda^2 - 4)(\lambda^2 - 9)$$
$$N_5 = \frac{4096}{105}\sigma^6(\lambda^2 - 4)(\lambda^2 - 9)(\lambda^2 - 16)$$

- The nonlinear terms are

$$\Lambda_n^{(4)} = \sum : L_{n-p} L_p :$$
$$\Lambda_n^{(5)} = \sum : L_{n-p} W_p :$$
$$\Lambda_n^{(6)} = \sum : W_{n-p} W_p :$$

AdS_3/CFT_2: [Gaberdiel & Gopakumar '10]

AdS_3

massless HS fields
$+2$ complex scalars
with equal mass

CFT_2

WZW coset model:

$$\frac{su(N)_k \oplus su(N)_1}{su(N+1)_{k+1}}$$

- 't Hooft limit: $N, k \to \infty$, $0 \leq \lambda \equiv \frac{N}{N+k} \leq 1$ fixed
- Central charge:

$$c = (N-1)\Big[1 - \frac{N(N+1)}{(N+k)(N+k+1)}\Big] \sim N$$

- Mass of the scalar fields: $M^2 = \lambda^2 - 1$
- Two special cases: $\lambda = 0$: free fermion; $\lambda = 1$: free boson.

Chern-Simons Formulation of Higher Spin Gravity in AdS_3

- The action [Blencowe '89]: $S = S_{cs}[A] - S_{cs}[\bar{A}]$ where

$$S_{cs}[A] = \frac{k_{cs}}{4\pi} \int \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

- The equations of motion:

$$\begin{aligned} F =& dA + A \wedge A = 0 \\ \bar{F} =& d\bar{A} + \bar{A} \wedge \bar{A} = 0 \end{aligned}$$

- The gauge fields:

$$\begin{aligned} A =& \omega + e \\ \bar{A} =& \omega - e \end{aligned}$$

where e is the veilbein and ω is the spin connection.

$\mathcal{W}$-symmetry as asymptotic symmetry

- The central charge [Brown & Henneaux '86]:

$$c = 6k_{cs} = \frac{3\ell}{2G_N}, \qquad k_{cs} = \frac{\ell}{4G_N}$$

where ℓ is the radius of the AdS space.

- Asymptotic symmetry: $hs[\lambda] \to \mathcal{W}_\infty[\lambda]$

Henneaux & Rey: 1008.4579
Campoleoni, Fredenhagen, Pfenninger & Theisen: 1008.4744
Gaberdiel & Hartman: 1101.2910
Campoleoni, Fredenhagen & Pfenninger: 1107.0290

BTZ black holes

- The metric:

$$ds^2 = d\rho^2 + \frac{2\pi}{k}\left(\mathcal{L}(dx^+)^2 + \bar{\mathcal{L}}(dx^-)^2\right) - \left(e^{2\rho} + \frac{4\pi^2}{k^2}\mathcal{L}\bar{\mathcal{L}}e^{-2\rho}\right)dx^+dx^-$$

where $x^\pm = t \pm \phi$, $\phi \cong \phi + 2\pi$ and

$$\mathcal{L} = \frac{M-J}{4\pi}, \qquad \bar{\mathcal{L}} = \frac{M+J}{4\pi}$$

with M the mass and J the angular momentum.

- In terms of the connections:

$$\begin{aligned} A &= (e^\rho L_1 - \frac{2\pi}{k}e^{-\rho}\mathcal{L}L_{-1})dx^+ + L_0 d\rho \\ \bar{A} &= -(e^\rho L_{-1} - \frac{2\pi}{k}\bar{\mathcal{L}}e^{-\rho}L_1)dx^- - L_0 d\rho \end{aligned}$$

where $L_{0,\pm 1}$ are the SL(2) generators.

Higher Spin Black Holes

- $SL(3)$: [Gutperle & Kraus '11]

$$A = L_0 d\rho + (e^\rho L_1 - \frac{2\pi}{k}\mathcal{L}e^{-\rho}L_{-1} + \frac{\pi}{2k\sigma}\mathcal{W}e^{-2\rho}W_{-2})dx^+$$
$$+ \frac{\alpha}{\bar{\tau}}(e^{2\rho}W_2 - \frac{4\pi}{k}\mathcal{L}W_0 + \frac{4\pi^2}{k^2}\mathcal{L}^2 e^{-2\rho}W_{-2} + \frac{4\pi}{k}\mathcal{W}e^{-\rho}L_{-1})dx^-$$

where α is the chemical potential of the spin-3 current.

- $hs[\lambda]$: [Kraus & Perlmutter '11]

$$A = b^{-1}ab + b^{-1}db, \qquad b = e^{\rho V_0^2}$$
$$a_+ = V_1^2 - \frac{2\pi\mathcal{L}}{k} - N(\lambda)\frac{\pi\mathcal{W}}{2k}V_{-2}^3 + J$$
$$a_- = \frac{\alpha}{\bar{\tau}}N(\lambda)\Big(a_+ * a_+ - \frac{2\pi\mathcal{L}}{3k}(\lambda^2 - 1)\Big)$$

where $N(\lambda)$ is a normalization factor and J contains infinite higher-spin fields: $J = J_4 V_{-3}^4 + J_5 V_{-4}^5 + \cdots$

The partition function

- Smoothness of the Euclidean horizon $\Leftrightarrow$ Holonomy conditions:

$$\mathrm{Tr}(w^n) = \mathrm{Tr}(w^n_{BTZ}), \qquad n = 2, 3, \cdots$$

where w is the holonomy matrix

$$w = 2\pi(\tau A_+ - \bar{\tau} A_-) = 2\pi\left[\tau a_+ - \alpha N(\lambda)\left(a_+ * a_+ - \frac{2\pi\mathcal{L}}{3k}(\lambda^2 - 1)\right)\right]$$

- Integrability condition: first law of thermodynamics

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{\partial \mathcal{W}}{\partial \tau}$$

- Calculation of the free energy:

$$\mathcal{L} = \langle \hat{\mathcal{L}} \rangle = -\frac{i}{4\pi^2}\frac{\partial \ln Z}{\partial \tau}, \qquad \mathcal{W} = \langle \hat{\mathcal{W}} \rangle = -\frac{i}{4\pi^2}\frac{\partial \ln Z}{\partial \alpha}$$

The gravity result:

- Free energy: [Kraus & Perlmutter '11]

$$\ln Z_{BH}(\hat{\tau},\alpha,\hat{\bar{\tau}},\bar{\alpha}) = \frac{i\pi c}{12\hat{\tau}}\Bigg[1-\frac{4}{3}\frac{\alpha^2}{\hat{\tau}^4}+\frac{400}{27}\frac{\lambda^2-7}{\lambda^2-4}\frac{\alpha^4}{\hat{\tau}^8} -\frac{1600}{27}\frac{5\lambda^4-85\lambda^2+377}{(\lambda^2-4)^2}\frac{\alpha^6}{\hat{\tau}^{12}} +\frac{32000}{81}\frac{20\lambda^6-600\lambda^4+6387\lambda^2-23357}{(\lambda^2-4)^3}\frac{\alpha^8}{\hat{\tau}^{16}}+\cdots\Bigg] + \text{rightmoving}$$

- From the CFT point of view:

$$\begin{aligned} Z(\hat{\tau},\hat{\bar{\tau}},\alpha,\bar{\alpha}) &= \mathrm{Tr}_{AdS}\left(e^{4\pi^2 i(\hat{\tau}\hat{\mathcal{L}}+\alpha\hat{\mathcal{W}}-\hat{\bar{\tau}}\hat{\bar{\mathcal{L}}}-\bar{\alpha}\hat{\bar{\mathcal{W}}})}\right) \\ &= \mathrm{Tr}_{CFT}\left(\hat{q}^{L_0-\frac{c}{24}}y^{W_0}\hat{\bar{q}}^{\bar{L}_0-\frac{c}{24}}\bar{y}^{\bar{W}_0}\right) \end{aligned}$$

where $\hat{q}=e^{2\pi i\hat{\tau}}$, $y=e^{2\pi i\alpha}$.

The problem:

- Under S-transformation: $\hat{\tau} = -1/\tau$, $q \to 0$

$$\mathrm{Tr}_i(\hat{q}^{L_0 - \frac{c}{24}} y^{W_0}) = \sum_j S_{ij} \cdots \mathrm{Tr}_j(\hat{q}^{L_0 - \frac{c}{24}} \cdots)$$

- The strategy:

$$\begin{aligned} Z_{CFT}(\hat{\tau}, \alpha) =& \mathrm{Tr}(\hat{q}^{L_0 - \frac{c}{24}} y^{W_0}) \\ =& \mathrm{Tr}\left(\hat{q}^{L_0 - \frac{c}{24}}\right) + \frac{(2\pi i)^2 \alpha^2}{2!} \mathrm{Tr}\left(W_0^2 \hat{q}^{L_0 - \frac{c}{24}}\right) \\ &+ \frac{(2\pi i)^4 \alpha^4}{4!} \mathrm{Tr}\left(W_0^4 \hat{q}^{L_0 - \frac{c}{24}}\right) + \cdots \end{aligned}$$

- The α-independent term:

$$\mathrm{Tr}\left(\hat{q}^{L_0 - \frac{c}{24}}\right) \to \frac{i\pi c}{12\hat{\tau}}$$

- Question: How traces with zero mode insertions behave under the modular transformation?

Toy model: the Jacobi forms

- The Jacobi forms are defined by

$$\phi_i(\tau, z) \equiv \mathrm{Tr}_i(y^{J_0} q^{L_0 - \frac{c}{24}}) \ , \qquad y = e^{2\pi i z} \ , \quad q = e^{2\pi i \tau}$$

  where the current J has conformal dimension 1.
- Under the modular transformation

$$\phi_i\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = \exp\left\{2\pi i k \frac{cz^2}{c\tau + d}\right\} \sum_j M_{ij}\, \phi_j(\tau, z)$$

  where the sum runs over all irreducible representations of the chiral algebra, and M_{ij} defines a representation of the modular group.
- The level k is the constant that appears in the commutator

$$[J_m, J_n] = 2km\delta_{m+n,0}$$

Expansion at $z = 0$:

- Expanding out the transformation rule in orders of z, the z^0, z^1 terms are the usual modular transformation rule of the characters.
- Higher orders:

$$\mathrm{Tr}_i(J_0 J_0\,\hat{q}^{L_0-\frac{c}{24}}) = \sum_j S_{ij}\Big[\tau^2\,\mathrm{Tr}_j(J_0 J_0\,q^{L_0-\frac{c}{24}}) + \frac{k}{\pi i}\,\tau\,\mathrm{Tr}_j(q^{L_0-\frac{c}{24}})\Big]$$

$$\mathrm{Tr}_i(J_0 J_0 J_0\,\hat{q}^{L_0-\frac{c}{24}}) = \sum_j S_{ij}\Big[\tau^3\mathrm{Tr}_j(J_0 J_0 J_0\,q^{L_0-\frac{c}{24}}) + \frac{3k}{\pi i}\tau^2\,\mathrm{Tr}_j(J_0 q^{L_0-\frac{c}{24}})\Big]$$

$$\mathrm{Tr}_i(J_0 J_0 J_0 J_0\,\hat{q}^{L_0-\frac{c}{24}}) = \sum_j S_{ij}\Big[\tau^4\,\mathrm{Tr}_j(J_0 J_0 J_0 J_0\,q^{L_0-\frac{c}{24}}) + 6\tau^3\,\frac{k}{\pi i}\,\mathrm{Tr}_j(J_0 J_0\,q^{L_0-\frac{c}{24}}) + 3\tau^2\,\frac{k^2}{(\pi i)^2}\,\mathrm{Tr}_j(q^{L_0-\frac{c}{24}})\Big]$$

- At high temperature: $q \to 0$, only the vacuum representation ($j = 0$) will contribute to the lowest order of the partition function $\Rightarrow$ No explicit zero modes after the S-transformation.

The comparison

- Under S-transformation:

$$\mathrm{Tr}(W_0 W_0\,\hat{q}^{L_0-\frac{c}{24}}) = \sum_i S_{i0}\Big[\#_2(\lambda,\tau)\mathrm{Tr}_0(q^{L_0-\frac{c}{24}})\Big]$$

$$\mathrm{Tr}(W_0 W_0 W_0 W_0\,\hat{q}^{L_0-\frac{c}{24}}) = \sum_i S_{i0}\Big[\#_4(\lambda,\tau)\mathrm{Tr}_0(q^{L_0-\frac{c}{24}})\Big]$$

- Collect the contributing terms

$$Z = \sum_i S_{i0}\Big[1 + \#_2(\lambda,\tau) + \#_4(\lambda,\tau) + \cdots\Big]q^{-\frac{c}{24}}$$

$$\sim q^{-\frac{c}{24}}\Big[1 + \#_2(\lambda,\tau) + \#_4(\lambda,\tau) + \cdots\Big]$$

- Exponentiating the gravity result

$$Z_{BH} = q^{-\frac{c}{24}}\Big[1 + \frac{i\pi c}{9}\alpha^2\tau^5 - \frac{100 i\pi c}{81}\frac{\lambda^2-7}{\lambda^2-4}\alpha^4\tau^9 + \cdots\Big]$$

Torus correlation functions

- The torus correlation functions are defined by

$$F((a^1, z_1), \ldots, (a^n, z_n); q) = z_1^{h_1} \cdots z_n^{h_n} \mathrm{Tr}\left(V(a^1, z_1) \cdots V(a^n, z_n) q^{L_0 - \frac{c}{24}}\right)$$

  where h_j are the conformal dimensions of the vertex operators $V(a^j, z_j)$.

- These functions are periodic under the transformations

$$z_j \mapsto e^{2\pi i} z_j, \qquad z_j \mapsto q z_j$$

  where the second period is proven using

$$V(a^j, q z_j) q^{L_0 - \frac{c}{24}} = q^{-h_j} q^{L_0 - \frac{c}{24}} V(a^j, z_j)$$

  and the cyclicity of the trace.

- Expanding the vertex operators $V(a, z) = \sum a_m z^{-m-h}$, the zero modes can be extracted via the contour integrals

$$\mathrm{Tr}(a_0^1 \cdots a_0^n q^{L_0 - \frac{c}{24}}) = \frac{1}{(2\pi i)^n} \oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_n}{z_n} F((a^1, z_1), \ldots, (a^n, z_n); q)$$

Modular transformation of the torus amplitude

- Under a modular transformation, the functions F_i transform as

$$F_i\Big((a^1,z_1),\ldots,(a^n,z_n);\frac{a\tau+b}{c\tau+d}\Big)=(c\tau+d)^{\sum_l h_l}$$
$$\times\sum_j M_{ij}F_j\left((a^1,z_1^{c\tau+d}),\ldots,(a^n,z_n^{c\tau+d});\tau\right)$$

  where $M_{ij}\equiv M_{ij}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is a representation of the modular group, i.e. a constant matrix for each modular transformation.

- In particular, for the S-transformation $\tau\mapsto -1/\tau$, we have $c=1$, $d=0$

$$F_i\Big((a^1,z_1),\ldots,(a^n,z_n);-\frac{1}{\tau}\Big)=\tau^{\sum_l h_l}\sum_j S_{ij}\,F_j((a^1,z_1^{\tau}),\ldots,(a^n,z_n^{\tau});\tau)$$

Extraction of the zero modes

- The zero modes can be extracted via the contour integrals

$$\mathrm{Tr}_r(a_0^1 \cdots a_0^n \hat{q}^{L_0 - \frac{c}{24}}) = \frac{1}{(2\pi i)^n} \oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_n}{z_n}$$
$$\tau^{\sum_l h_l} \sum_s S_{rs} F_s((a^1, z_1^\tau), \ldots, (a^n, z_n^\tau); \tau)$$

- After a change of variables:

$$\frac{1}{2\pi i} \oint \frac{dz}{z} = \int_0^1 d\theta = \frac{1}{\tau} \int_0^\tau d(\tau\theta) = \frac{1}{2\pi i\tau} \int_1^q \frac{d\tilde{z}}{\tilde{z}}$$

with $z = e^{2\pi i\theta}$ and $\tilde{z} \equiv z^\tau = e^{2\pi i\tau\theta}$.

- The final formula:

$$\mathrm{Tr}_r(a_0^1 \cdots a_0^n \hat{q}^{L_0 - \frac{c}{24}}) = \frac{1}{(2\pi i)^n} \tau^{-n+\sum_j h_j} \sum_s S_{rs}$$
$$\int_1^q \frac{d\tilde{z}_1}{\tilde{z}_1} \cdots \int_1^q \frac{d\tilde{z}_n}{\tilde{z}_n} F_s((a^1, \tilde{z}_1), \ldots, (a^n, \tilde{z}_n); \tau)$$

Torus recursion relations: [Zhu '96]

$$F\big((a^1, z_1), (a^2, z_2), \ldots, (a^n, z_n); q\big) = F\big(a^1_0, (a^2, z_2), \ldots, (a^n, z_n); q\big)$$
$$+ \sum_{j=2}^{n} \sum_{m=0}^{\infty} \mathcal{P}_{m+1}\left(\frac{z_j}{z_1}, q\right) \times F\big((a^2, z_2), \ldots, (a^1[m]a^j, z_j), \ldots, (a^n, z_n); q\big)$$

where $\mathcal{P}$ is the Weierstrass function and the bracketed modes are defined via

$$a[m] = (2\pi i)^{-m-1} \sum_{i \geq m} c(h_a, i, m) a_{-h_a+1+i}$$

The coefficients $c(h_a, i, m)$ are found by the expansion:

$$(\ln(1+z))^n (1+z)^{h-1} = \sum_{j \geq n} c(h, j, n) z^j.$$

For insertion of W fields: $h = 3$

$$W[1] = (2\pi i)^{-2} \left(W_{-1} + \frac{3}{2} W_0 + \frac{1}{3} W_1 - \frac{1}{12} W_2 + \frac{1}{30} W_3 + \cdots \right)$$

The Weierstrass function

The Weierstrass functions are defined by the power series

$$\mathcal{P}_k(x,q) = \frac{(2\pi i)^k}{(k-1)!} \sum_{m\neq 0} \left(\frac{m^{k-1} x^m}{1-q^m} \right), \qquad k \geq 1$$

They satisfy the important recursion relation

$$x \frac{d}{dx} \mathcal{P}_k(x,q) = \frac{k}{2\pi i} \mathcal{P}_{k+1}(x,q)$$

Periodicity: $x \to qx$

$$\mathcal{P}_1(qx,q) = \mathcal{P}_1(x,q) + 2\pi i, \qquad \mathcal{P}_k(qx,q) = \mathcal{P}_k(x,q) \qquad (k>1)$$

Integral:

$$\int_1^q \frac{dz_2}{z_2} \mathcal{P}_2\left(\frac{z_1}{z_2}, q\right) = (2\pi i)^2, \quad \int_1^q \frac{dz_2}{z_2} \mathcal{P}_{m+1}\left(\frac{z_1}{z_2}, q\right) = 0 \quad (m>1)$$

The two-point function

$$Z^{(2)} \equiv \frac{(2\pi i\alpha)^2}{2!}\,\mathrm{Tr}\big(W_0 W_0 \hat{q}^{L_0-\frac{c}{24}}\big)$$
$$\approx \frac{\alpha^2\tau^4}{2}\int_1^q \frac{dz_1}{z_1}\int_1^q \frac{dz_2}{z_2} F\big((W,z_1),(W,z_2);\tau\big)$$

Applying the recursion relation, we find

$$F\big((W,z_1),(W,z_2);\tau\big) = z_2^3\,\mathrm{Tr}\big(W_0 W(z_2)\, q^{L_0-\frac{c}{24}}\big)$$
$$+\sum_m \mathcal{P}_{m+1}\left(\frac{z_2}{z_1}\right) F\big((W[m]W,z_2);\tau\big)$$

Only the $m=1$ term will contribute $\quad W(z)=V(W_{-3}\Omega,z),\ V(\Omega,z)=1$

$$Z^{(2)} \approx \frac{1}{2}q^{-\frac{c}{24}}(2\pi i)^3\alpha^2\tau^5\langle W[1]W_{-3}\rangle \approx \frac{1}{2}q^{-\frac{c}{24}}(2\pi i)\alpha^2\tau^5\frac{1}{30}\langle W_3 W_{-3}\rangle$$

The central charge term:

$$[W_3, W_{-3}] \sim \frac{5N_3 c}{6} \Rightarrow Z^{(2)} \approx \frac{i\pi c}{36}N_3\alpha^2\tau^5 q^{-\frac{c}{24}}$$

Normalization

- The constant

$$N_3 = \frac{16}{5}\sigma^2(\lambda^2 - 4)$$

- Using the WW OPE

$$W(z)W(0) \sim \frac{10c}{3}\frac{1}{z^6} + \cdots$$

- The normalization constant

$$\sigma^2 = \frac{5}{4(\lambda^2 - 4)} \Rightarrow N_3 = 4$$

- The agreement of the two-point result

$$Z^{(2)} \approx \frac{i\pi c}{9}\alpha^2\tau^5 q^{-c/24}$$

The four-point case

$$Z^{(4)} \equiv \frac{(2\pi i\alpha)^4}{4!}\,\mathrm{Tr}\big(W_0 W_0 W_0 W_0 \hat{q}^{L_0-\frac{c}{24}}\big)$$
$$\approx \frac{\alpha^4\tau^8}{4!}\int F\big((W,z_1),(W,z_2),(W,z_3),(W,z_4);\tau\big)$$

Applying the recursion relation once, we get

$$\int F\big((W,z_1),\ldots,(W,z_4);\tau\big) =$$
$$\int F\big(W_0;(W,z_2),(W,z_3),(W,z_4);\tau\big)$$
$$+\,3\int \mathcal{P}_{\ell+1}\left(\frac{z_4}{z_1}\right) F\big((W,z_2),(W,z_3),(W[\ell]W,z_4);\tau\big)$$

What about the zero mode term?

$$[W_0, W_m] = -2mU_m - \frac{N_3}{6}m(m^2-4)L_m + \cdots$$

Zero mode recursion relations: [Gaberdiel, Hartman & KJ '12]

$$F\big(b_0^\ell;(a^1,z_1),\ldots,(a^n,z_n);\tau\big) = z_1^{h_1}\cdots z_n^{h_n}\ \mathrm{Tr}\Big(b_0^\ell\, V(a^1,z_1)\ldots V(a^n,z_n)\, q^{L_0-\frac{c}{24}}\Big)$$

The recursion relation:

$$F\big(b_0^\ell;(a^1,z_1),\ldots,(a^n,z_n);\tau\big) = F\left(b_0^\ell a_0^1;(a^2,z_2),\ldots,(a^n,z_n);\tau\right)$$
$$+\sum_{i=0}^{\ell}\sum_{j=2}^{n}\sum_{m\in\mathbb{N}_0}\binom{\ell}{i}g_{m+1}^i\left(\frac{z_j}{z_1}\right)$$
$$\times F\left(b_0^{\ell-i};(a^2,z_2),\ldots,(d^{(i)}[m]a^j,z_j),\ldots,(a^n,z_n);\tau\right)$$

where

$$g_{m+1}^i(x,q) = (2\pi i)^i\frac{(m-i)!}{m!}\partial_\tau^i\mathcal{P}_{m+1-i}(x,q) \qquad (m\geq i)$$

and

$$d^{(i)} = (-1)^i(b[0])^i a^1$$

Applying the recursion relation

$$
\begin{aligned}
&\int F\big((W,z_1),\ldots,(W,z_4);\tau\big) = \\
&\quad 3\int \mathcal{P}_{\ell+1}\left(\frac{z_4}{z_1}\right)\mathcal{P}_{m+1}\left(\frac{z_4}{z_2}\right)\mathcal{P}_{k+1}\left(\frac{z_4}{z_3}\right)\langle W[k]W[m]W[\ell]W_{-3}\rangle \\
&+3\int \mathcal{P}_{\ell+1}\left(\frac{z_4}{z_1}\right)\mathcal{P}_{m+1}\left(\frac{z_3}{z_2}\right)\mathcal{P}_{k+1}\left(\frac{z_4}{z_3}\right)\langle (W[m]W)[k]W[\ell]W_{-3}\rangle \\
&+\frac{5(2\pi i)}{m}\int \mathcal{P}_{\ell+1}\left(\frac{z_4}{z_1}\right)\partial_\tau\mathcal{P}_{m}\left(\frac{z_4}{z_3}\right)\langle d^{(1)}[m]W[\ell]W_{-3}\rangle \\
&+\frac{2(2\pi i)}{\ell}\int \partial_\tau\mathcal{P}_{\ell}\left(\frac{z_4}{z_2}\right)\mathcal{P}_{m+1}\left(\frac{z_4}{z_3}\right)\langle W[m]d^{(1)}[\ell]W_{-3}\rangle \\
&+\frac{(2\pi i)^2}{\ell(\ell-1)}\int \partial_\tau^2\mathcal{P}_{\ell-1}\left(\frac{z_4}{z_3}\right)\langle d^{(2)}[\ell]W_{-3}\rangle
\end{aligned}
$$

where we have defined the states

$$
d^{(1)} \equiv -W[0]W\ , \qquad \text{and} \qquad d^{(2)} \equiv W[0]W[0]W
$$

The four-point result

- In each expectation value, only the bracket modes sum to 3 will contribute at leading c
- The nested modes can be expanded using the identity [Zhu '96]

$$(a[m]b)[n] = \sum_{i=0}^{m} \binom{m}{i} \Big((-1)^i a[m-i]b[n+i] - (-1)^{m+i} b[m+n-i]a[i] \Big)$$

- The final result:

$$\begin{aligned} Z^{(4)} &\approx -q^{-c/24}\, 2\pi i c\, \frac{2}{27} \left(5N_3^2 - 7N_4\right) \alpha^4 \tau^9 \\ &\approx -q^{-c/24}\, \frac{100 i \pi c}{81} \frac{\lambda^2 - 7}{\lambda^2 - 4} \alpha^4 \tau^9 \end{aligned}$$

agrees with the gravity result.

Nonlinear contribution to the six-point function

Schematically, the spin-3 commutators of $\mathcal{W}_\infty[\lambda]$ have the form

$$[W, W] \sim U + L + \frac{1}{c}\Lambda^{(4)} + c$$

where U is the spin-4 current.

In the six-point case, we can contract two currents to make L and two other currents to make U

$$\underbrace{WW}\,\underbrace{WW}WW \to L\,U\,W\,W$$

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Now contracting $\underbrace{UW} \sim \frac{1}{c}LW + X + W$ gives the nonlinear term

$$\frac{1}{c}\,L\,L\,W\,W$$

Since LL and WW both have central terms, this term is of order c.

The six-point result

- All the rest calculations are similar as the four-point function
- There are 37 nonzero contractions of the form:

$$\langle W[i]W[j]W[k]W[l]W[m]W_{-3}\rangle$$

satisfying $i+j+k+l+m=5$

- One example is like:

$$\langle W[1]^5 W_{-3}\rangle = (2\pi i)^{-10} c \left(\frac{10N_3^3}{9} - \frac{364 N_3 N_4}{135} + \frac{2704 N_4^2}{2025 N_3} + \frac{179 N_5}{63}\right)$$

- The final result:

$$\begin{aligned} Z^{(6)} &\approx q^{-c/24}\, 2\pi i c \left(\frac{17N_3^3}{648} - \frac{581 N_3 N_4}{9720} + \frac{497 N_4^2}{12150 N_3} + \frac{101 N_5}{2160}\right)\alpha^6\tau^{13} \\ &\approx q^{-c/24}\, \frac{400 i\pi c}{81}\, \frac{5\lambda^4 - 85\lambda^2 + 377}{(\lambda^2-4)^2}\alpha^6\tau^{13} \end{aligned}$$

agrees with the gravity result.

Summary

- We reproduced the higher spin corrections to the black hole entropy from calculating correlation functions of $\mathcal{W}$-currents on the torus
- The calculation depends on the full nonlinear $\mathcal{W}_\infty[\lambda]$ structure
- This gives a detailed/different check that $\mathcal{W}_\infty[\lambda]$ is indeed the correct symmetry algebra of the dual CFT
- Our method also applies to black hole solutions with more chemical potentials