Higher spin theories, holography and Chern-Simons vector models

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Outline

- The Klebanov-Polyakov-Sezgin-Sundell conjectures:
  - HS gravity in $AdS_4 \quad \leftrightarrow \quad$ 3d vector models
- Structure of Vasiliev's higher spin gauge theory in 4d
  - the "Type A" and "Type B" models
  - Parity violating models
- Testing the KPSS conjecture: the three-point functions
- Chern-Simons theory with vector fermion matter
  - Exact thermal free energy on R^2
  - Higher spin symmetry at large N and conjectural AdS dual
- Summary and conclusions

Based on arXiv:0912.3462, arXiv:1004.3736, arXiv:1105.4011 (SG, Yin)
arXiv:1104.4317 (SG, Prakash, Yin)

and arXiv:1110.4386 (SG, Minwalla, Prakash, Trivedi, Wadia, Yin)

The free $O(N)$ vector model

In 3d, consider an *N-vector* of real scalar fields ϕ^i with free action

$$S_0 = \frac{1}{2}\int d^3x \sum_{i=1}^{N} (\partial_\mu \phi^i)^2$$

and impose a restriction to the sector of $O(N)$ singlet operators.

The free theory has an infinite tower of conserved *higher spin currents*

$$J_{\mu_1\cdots\mu_s} = \phi^i \partial_{(\mu_1} \cdots \partial_{\mu_s)} \phi^i + \ldots \qquad s = 2, 4, 6, \ldots$$
$$\partial^\mu J_{\mu\mu_2\cdots\mu_s} = 0 \qquad \Delta(J_s) = s + 1$$

Together with the scalar operator

$$J_0 = \phi^i \phi^i \qquad \Delta(J_0) = 1$$

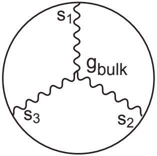
these are all the "single-trace" (single sum over i) primaries. They should be dual to single particle states in the bulk dual.

The free $O(N)$ vector model

- By the standard rules of holography, the conserved HS currents J_s should be dual to *higher spin gauge fields* in AdS.
- For normalized currents:

$$\langle J_s J_s \rangle \sim N^0 \qquad \langle J_{s_1} J_{s_2} J_{s_3} \rangle \sim N^{-1/2}$$

indicating that the bulk coupling constant is $g_{\rm bulk} \sim \frac{1}{\sqrt{N}}$.



- Hence the dual should be a theory of interacting higher spin *gauge fields* in AdS.

- Fully non-linear consistent theories of interacting higher spin gauge fields indeed exist in (A)dS, as discovered by Vasiliev ('86-'03).
- For this talk: Vasiliev's bosonic HS gauge theory on AdS_4
  - Contains a scalar plus an infinite tower of HS gauge fields, one for each integer spin. Includes gravity.
  - Admits a consistent truncation to a "minimal" theory with even spins only.
  - Essentially unique structure (up to a choice of "interaction phase"). Requiring a parity symmetry yields only 2 allowed models: "type A" (parity even scalar field) and "type B" (parity odd scalar)
  - Interactions carry arbitrarily high derivatives. Non-local.
  - Might be a UV finite 4d quantum gravity theory?

Klebanov-Polyakov-Sezgin-Sundell ('02) conjecture:

Vasiliev's minimal bosonic (type A) HS gauge theory in AdS_4 is dual to free/critical *3d* $O(N)$ *vector model*, in the $O(N)$ singlet sector.

- Why vector models? A free gauge theory of SYM type also has HS cunserved currents $J_s \sim \mathrm{Tr}\Phi\partial^s\Phi$. But in addition there are many more single trace operators $\mathrm{Tr}\,\Phi\partial^{k_1}\Phi\partial^{k_2}\Phi\cdots\partial^{k_n}\Phi$, which should be dual to massive fields in the bulk.
- In a vector theory, operators of the form $(\phi^i\partial\cdots\partial\phi^i)(\phi^j\partial\cdots\partial\phi^j)$ are analogous to multi-trace operators and should be thought as multi-particle states from bulk point of view.
- A vector model has precisely the right spectrum to be dual to a *pure* HS gauge theory!

Klebanov-Polyakov-Sezgin-Sundell ('02) conjecture:

Vasiliev's minimal bosonic (type A) HS gauge theory in AdS_4 is dual to free/critical *3d* $O(N)$ *vector model*, in the $O(N)$ singlet sector.

- The restriction to singlet sector is important to match boundary and bulk spectrum. It may be implemented by gauging the $O(N)$ symmetry and taking a limit of zero gauge coupling. In practice, we may couple the vector field to a Chern-Simons gauge field at level k, and take the limit $k \to \infty$.

Klebanov-Polyakov-Sezgin-Sundell ('02) conjecture:

Vasiliev's minimal bosonic (type A) HS gauge theory in AdS_4 is dual to free/critical *3d* $O(N)$ *vector model*, in the $O(N)$ singlet sector.

- Critical $O(N)$ model: It is the IR fixed point of a relevant $\lambda(\phi^i\phi^i)^2$ deformation of the free theory. At the critical point, the spectrum of single trace primaries is

$$J_0\,, \quad \Delta_0 = 2 + \mathcal{O}(1/N) \qquad J_s\,, \quad \Delta_s = s + 1 + \mathcal{O}(1/N)$$

  The HS currents are conserved at $N = \infty$. HS broken by $1/N$ effects (anomalous dimensions). Loop effect from bulk viewpoint (self-energy diagram of HS field).
- An *interacting* CFT_3 that should be dual to a HS gauge theory. It does not contradict Maldacena-Zhiboedov's theorem, because the HS symmetry is broken $\partial \cdot J_s \sim \frac{1}{\sqrt{N}} \sum_{s'} \partial J_{s'} \partial J_0$.

Free vs Critical $O(N)$ model from the bulk

How can the same bulk theory be dual to two different CFT's?

It turns out that the Vasiliev's bulk scalar φ has $m^2 = -2/R_{AdS}^2$. Then by the AdS/CFT dictionary $\Delta(\Delta - d) = m^2$, both solutions $\Delta = 1$ or $\Delta = 2$ are acceptable.

Two inequivalent choices of boundary conditions: $\varphi(z, \vec{x}) \to z^{\Delta}\varphi_0(\vec{x})$ as $z \to 0$, with $\Delta = 1$ or $\Delta = 2$.

- $\Delta = 1$ boundary condition $\rightarrow$ dual to *free* $O(N)$ model.
- $\Delta = 2$ boundary condition $\rightarrow$ dual to *critical* $O(N)$ model.

It is the vector analogue of the general story about relevant double-trace deformations ("double-trace" deformation here is $\lambda(\phi^i\phi^i)^2$.)

The conjecture for the type B model (Sezgin, Sundell '03)

- There is a natural generalization of the conjecture for the "type B" Vasiliev's HS theory which has a parity odd bulk scalar. Again one has two possible dual CFT's depending on the choice of boundary condition for the scalar.
- $\Delta = 2$ boundary condition $\rightarrow$ dual to *free N-fermion* theory in the $O(N)$ singlet sector

$$S = \int d^3x \, \psi^i \gamma^\mu \partial_\mu \psi^i \qquad i = 1, \ldots, N$$

"Single-trace" operators:

$$J_0 = \psi^i \psi^i, \quad \underline{\Delta_0 = 2} \qquad J_s \sim \psi^i \gamma_{(\mu_1} \partial_{\mu_2} \cdots \partial_{\mu_s)} \psi^i, \quad \Delta_s = s + 1$$

J_0 is dual to the parity odd bulk scalar with $\Delta = 2$ b.c.

- $\Delta = 1$ boundary condition $\rightarrow$ dual to the interacting fixed point of the N-fermion theory perturbed by quartic interaction.

Comment on "non-minimal" HS theories

- All these conjectures can be naturally generalized to the case of so-called "non-minimal" bosonic HS theory which include all the integer spins $s = 0, 1, 2, 3 \dots$
- Instead of $O(N)$ real scalars/fermions, one considers theories of *complex* scalar or fermions, restricted to $U(N)$ singlet sector.

$$S = \int d^3x \, \partial^\mu \bar{\phi}_i \partial_\mu \phi^i$$

The odd-spin currents are now non-trivial, e.g. $J_\mu^{(1)} = \bar{\phi}_i \overleftrightarrow{\partial}_\mu \phi^i$ etc, and they are dual to the HS gauge fields of odd spins.

- In the following I will not assume truncation to the "minimal" theories.

Testing the conjectures

- In the free limit in the bulk, tests of these conjectures amount to matching the spectrum of bulk one-particle states with the boundary "single-trace" operators, which indeed agree.
- Evidence at the interacting level?
- Our aim is to use Vasiliev's non-linear theory to compute holographically the 3-point functions $\langle J_{s_1}(x_1)J_{s_2}(x_2)J_{s_3}(x_3)\rangle$ for general spins, and compare to vector models at the boundary.
- Conformal invariance and conservation do not completely fix the three-point functions.

- For example, for the stress tensor ($s = 2$), the 3-point function $\langle TTT \rangle$ in 3d is constrained to be a linear combination of 2 parity even structures (Osborn, Petkou '94), which are realized by free scalar and free fermion theories, plus one additional parity odd structure (SG, Prakash, Yin '11, Maldacena, Pimentel '11)

$$\langle TTT \rangle = a_1 \langle TTT \rangle_B + a_2 \langle TTT \rangle_F + b \langle TTT \rangle_{\text{parity odd}}$$

  the parity odd structure can arise in parity violating theories (which are necessarily *interacting* CFT's).
- The cubic vertex of Einstein gravity yields a linear combination of "B" and "F" structures. Vasiliev's theory must have the precise higher derivative structure to produce $a_2 = 0/a_1 = 0$ in type A/type B models.
- For conserved higher spin currents $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$, there is an analogous decomposition in 3 tensor structures (SG, Prakash, Yin '11, Costa et al '11, Maldacena, Zhiboedov '11). The parity odd structure exists only when $|s_2 - s_3| \leq s_1 \leq s_2 + s_3$.

Introducing Vasiliev's HS gauge theory

- Variables:
  1. x^μ: space-time coordinates
  2. $(Y, Z) \equiv (y_\alpha, \bar{y}_{\dot\alpha}, z_\alpha, \bar{z}_{\dot\alpha})$: *auxiliary twistor variables* $\quad \alpha, \dot\alpha = 1, 2$

  Commuting 2-comp. spinors, e.g. $y^\alpha y_\alpha = \epsilon^{\alpha\beta} y_\alpha y_\beta = 0$

  Y, Z-space endowed with a *star-product*:

  $$f(Y,Z) * g(Y,Z) = \int d^4U d^4V\, e^{u^\alpha v_\alpha + \bar{u}^{\dot\alpha} \bar{v}_{\dot\alpha}} f(Y+U, Z+U) g(Y+V, Z-V)$$

  in particular $y_\alpha * y_\beta = y_\alpha y_\beta + \epsilon_{\alpha\beta}$, $y_\alpha * z_\beta = y_\alpha z_\beta - \epsilon_{\alpha\beta}$ etc. Bilinears generate $SO(3,2)$ under $*$-commutators.

- Master fields:
  1. $W(x|y, \bar{y}, z, \bar{z}) = W_\mu dx^\mu$ $\quad$ 1-form in space-time
  2. $S(x|y, \bar{y}, z, \bar{z}) = S_\alpha dz^\alpha + S_{\dot\alpha} d\bar{z}^{\dot\alpha}$ $\quad$ 1-form in Z-space
  3. $B(x|y, \bar{y}, z, \bar{z})$ $\quad$ scalar

- Expansion of the master fields in powers of the (y, z)-spinor variables contains the physical space-time fields and their derivatives.

The Vasiliev's equations

- Collecting W and S into the 1-form $\mathcal{A} = W_\mu dx^\mu + S_\alpha dz^\alpha + S_{\dot\alpha} d\bar{z}^{\dot\alpha}$, Vasiliev's equation can be written as

$$
\begin{aligned}
&d\mathcal{A} + \mathcal{A} * \mathcal{A} = \mathcal{V}(B * \kappa) dz^2 + \bar{\mathcal{V}}(B * \bar\kappa) d\bar{z}^2 \\
&dB + \mathcal{A} * B - B * \pi(\mathcal{A}) = 0
\end{aligned}
$$

$d = d_x + d_Z$
$\kappa = e^{z^\alpha y_\alpha}, \bar\kappa = e^{\bar{z}^{\dot\alpha} \bar{y}_{\dot\alpha}}$: "Kleinian". $\kappa * \kappa = 1$
$\pi(f(y, \bar{y}, z, \bar{z}, dz, d\bar{z})) = f(-y, \bar{y}, -z, \bar{z}, -dz, d\bar{z})$

- Gauge symmetry: $\delta\mathcal{A} = d\epsilon + [\mathcal{A}, \epsilon]_*$ $\qquad \epsilon = \epsilon(x|y, \bar{y}, z, \bar{z})$
  $\delta B = \epsilon * B - B * \pi(\epsilon)$ $\qquad$ "twisted adjoint"

  Expressed in terms of physical d.o.f., this gives the HS gauge symmetry. At non-linear level, it requires infinite tower of spins.

The Vasiliev's equations

$$d\mathcal{A} + \mathcal{A} * \mathcal{A} = \mathcal{V}(B * \kappa) dz^2 + \bar{\mathcal{V}}(B * \bar{\kappa}) d\bar{z}^2$$
$$dB + \mathcal{A} * B - B * \pi(\mathcal{A}) = 0$$

- By freedom of field redefinitions, $\mathcal{V}(X)$ can be put in the form

$$\mathcal{V}(X) = X\, exp_*(i\Theta(X))\,, \qquad \Theta(X) = \theta_0 + \theta_2 X * X + \ldots$$

A choice of $\Theta(X)$ characterizes the interactions in the theory.
(at the level of 3-point functions, only θ_0 actually enters. θ_2 starts entering in 5-point functions etc)

The type A and type B models

- If we impose that the theory has a parity symmetry ($\vec{x} \to -\vec{x}$, $y_\alpha, z_\alpha \leftrightarrow \bar{y}_{\dot\alpha}, \bar{z}_{\dot\alpha}$), this interaction freedom is reduced to only 2 inequivalent choices (*Vasiliev, Sezgin-Sundell*)

  - $\Theta(X) = 0, \quad \text{i.e.} \quad \mathcal{V}(X) = X \qquad \text{if} \quad B \quad \text{is parity even}$
  - $\Theta(X) = \frac{\pi}{2}, \quad \text{i.e.} \quad \mathcal{V}(X) = iX \qquad \text{if} \quad B \quad \text{is parity odd}$

  which correspond respectively to the "type A" and "type B" models, conjecturally dual to scalar/fermion vector models (free or critical).

- If we do not require parity symmetry, we have a larger class of possible parity breaking higher spin gravity theories parameterized by a choice of $\Theta(X)$.

$$
\begin{aligned}
&d\mathcal{A} + \mathcal{A} * \mathcal{A} = B * \kappa\, e^{i\Theta(B*\kappa)} dz^2 + B * \bar{\kappa}\, e^{-i\Theta(B*\bar{\kappa})} d\bar{z}^2 \\
&dB + \mathcal{A} * B - B * \pi(\mathcal{A}) = 0
\end{aligned}
$$

- At least classically, these are all consistent HS gauge theories in AdS_4. One may ask what are the dual CFTs. We will make a natural proposal later.

AdS_4 vacuum

- Vasiliev's equations admit the vacuum solution $B_0 = S_0 = 0$ and $W_0 = W_0(x|Y)$ given by

$$W_0(x|Y) = (e_0)_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} + (\omega_0)_{\alpha\beta} y^\alpha y^\beta + (\omega_0)_{\dot{\alpha}\dot{\beta}} \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}}$$

which satisfies $d_x W_0 + W_0 * W_0 = 0$ if e_0 and ω_0 are vielbein and spin connection of AdS_4. In Poincare coordinates:

$$e_0 = \frac{dx^\mu}{4z} \sigma^\mu_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}}, \quad \omega_0 = \frac{dx^i}{8z} \left[(\sigma^{iz})_{\alpha\beta} y^\alpha y^\beta + (\sigma^{iz})_{\dot{\alpha}\dot{\beta}} \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}} \right]$$

- Fluctuations around this vacuum can be studied perturbatively

$$W = W_0 + W_1 + W_2 + \ldots, \quad B = B_1 + B_2 + \ldots, \quad S = S_1 + S_2 + \ldots$$

At linearized level, the equations can be shown to describe propagation of free HS gauge fields in AdS_4 plus one scalar with $m^2 = -2$.

Holographic three-point functions

- We want to compute the general 3-point functions $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$ from Vasiliev's theory.
- It is convenient to work with the currents contracted with a *null* polarization vector

$$J_s(x;\varepsilon) = J_{\mu_1\cdots\mu_s}(x)\varepsilon^{\mu_1}\cdots\varepsilon^{\mu_s} \qquad \varepsilon^{\mu}\varepsilon_{\mu} = 0$$

- It will be also useful to trade the null polarization vector ϵ^{μ} with a 2-component "polarization spinor" λ_{α} by

$$\epsilon_{\alpha\beta} = \vec{\epsilon}\cdot\vec{\tau}_{\alpha\beta} = \lambda_{\alpha}\lambda_{\beta}$$

where $\vec{\tau}$ are the 3 Pauli matrices. In other words we can work with $J_s(x;\varepsilon)$ or equivalently with

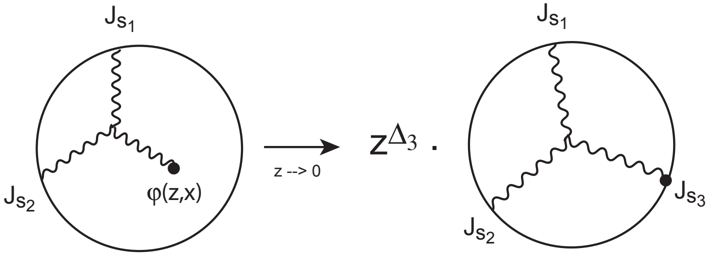
$$J_s(x;\lambda) = J_{\alpha_1\cdots\alpha_{2s}}(x)\lambda^{\alpha_1}\cdots\lambda^{\alpha_{2s}}$$

Holographic three-point functions

- We will extract the correlation functions directly from the equations of motion.
- We pick two of the currents, say J_{s_1}, J_{s_2}, to be sources, and we solve for the corresponding linearized bulk fields "$\varphi_{(1)}, \varphi_{(2)}$", i.e. we solve for the bulk-to-boundary propagators.
- Then we solve the e.o.m. for the second order fields, schematically:

$$D\varphi = \varphi_{(1)} * \varphi_{(2)}$$

- The 3-point function is then read off from the leading boundary behavior of the second order field φ.



$$\langle \varphi_{(s_3)}(z,\vec{x}_3)\rangle_{J_{s_1},J_{s_2}} \quad \xrightarrow{z\to 0} \quad z^{s_3+1}\,\langle J_{s_1}(\vec{x}_1)J_{s_2}(\vec{x}_2)J_{s_3}(\vec{x}_3)\rangle_{\mathrm{CFT}}$$

$$\Delta_s = s+1$$

3-point functions from Vasiliev's theory

- Solving Vasiliev's equation to second order in perturbation theory, we find that the 3-point functions of currents of all spins are encoded in the following integral over the internal twistor-like variables

$$\begin{aligned}
&\langle J(\vec{x}_1;\lambda_1)J(\vec{x}_2;\lambda_2)J(\vec{0};\lambda_3)\rangle \\
&= \int d^2u d^2\bar{u} d^2v d^2\bar{v}\, e^{uv-\bar{u}\bar{v}}\, B^{(1)}(u,\bar{u};\vec{x}_i,\lambda_i)B^{(1)}(v,\bar{v};\vec{x}_i,\lambda_i)\times \\
&\times \left[e^{2\lambda_3(u+v)}\delta(\bar{u}-\bar{v}+\sigma^z(u+v)) + e^{2\lambda_3(u-v)}\delta(\bar{u}+\bar{v}+\sigma^z(-u+v))\right]
\end{aligned}$$

  - $B^{(1)}(u,\bar{u};\vec{x}_i,\lambda_i) = e^{i\theta_0}B^{(1)}_+(u,\bar{u};\vec{x}_i,\lambda_i) + e^{-i\theta_0}B^{(1)}_-(u,\bar{u};\vec{x}_i,\lambda_i)$ are solutions of the linearized equations. They depend on positions and polarizations $\vec{x}_{1,2},\lambda_{1,2}$ of the two currents chosen as sources.
  - $B^{(1)}$ becomes extremely simple if we Fourier transform over the polarization spinors

$$\begin{aligned}
B^{(1)}_{tw}(y,\bar{y};\vec{x},\mu) &= \int d^2\lambda\, e^{\lambda^\alpha\mu_\alpha} B^{(1)}(y,\bar{y};\vec{x},\lambda) \\
&= e^{i\theta_0}\delta(y-\chi)e^{\bar{y}\bar{\chi}} + e^{-i\theta_0}\delta(\bar{y}-\bar{\chi})e^{y\chi} \qquad \chi=\chi(\vec{x})
\end{aligned}$$

Final result for the general 3-point function

- After performing the (U, V)-integration, we obtain the structure

$$\langle JJJ \rangle = \cos^2\theta_0 \langle JJJ \rangle_B + \sin^2\theta_0 \langle JJJ \rangle_F$$

where (re-inserting general position of third current, $\vec{0} \to \vec{x}_3$)

$$\langle JJJ \rangle_B = \frac{1}{|x_{12}||x_{23}||x_{31}|} \cosh\left[\frac{1}{2}(Q_1 + Q_2 + Q_3)\right] \cosh P_1 \cosh P_2 \cosh P_3$$

$$\langle JJJ \rangle_F = \frac{1}{|x_{12}||x_{23}||x_{31}|} \sinh\left[\frac{1}{2}(Q_1 + Q_2 + Q_3)\right] \sinh P_1 \sinh P_2 \sinh P_3$$

$$P_1 = \lambda_2 \frac{\mathbf{x_{23}}}{x_{23}^2} \lambda_3, \;\; Q_1 = \lambda_1 \frac{\mathbf{x_{12} x_{23} x_{13}}}{x_{12}^2 x_{31}^2} \lambda_1, \;\; \text{etc.} \qquad \mathbf{x_{ij}} = \vec{\tau} \cdot \vec{x}_{ij}$$

- It can be checked that these are precisely the generating functions of 3-point functions of HS currents of all spins in the free scalar and free fermion theories!
- Hence for $\theta_0 = 0$ (type A) and $\theta_0 = \pi/2$ (type B) we exactly confirm the conjectured duality to the bosonic and fermionic vector models!

Chern-Simons vector model

SG, S. Minwalla, S. Prakash, S. Trivedi, S. Wadia, X. Yin 2011

- Consider the 3d theory of a fundamental massless fermion coupled to a $U(N)$ Chern-Simons gauge field at level k

$$S = \frac{k}{4\pi} S_{CS}(A) + \int d^3x \, \bar{\psi}_i \gamma^\mu D_\mu \psi^i \qquad i = 1, \ldots, N$$

- In 3d, ψ has dimension 1, and the only marginal coupling is the Chern-Simons coupling k. This cannot run because it is quantized to be integer.
- Fine-tuning the mass of the fermion to zero, we obtain a family of interacting CFT's labelled by two integers k, N.
- Taking $k \to \infty$, this reduces to the singlet sector of the free fermionic vector model dual to Vasiliev's type B theory.

Chern-Simons vector model

$$S = \frac{k}{4\pi} S_{CS}(A) + \int d^3x\, \bar{\psi}_i \gamma^\mu D_\mu \psi^i \qquad\qquad i = 1, \ldots, N$$

- We will be interested in the large N 't Hooft limit

$$N \to \infty, k \to \infty \qquad \text{with} \;\; \lambda = \frac{N}{k} \;\; \text{fixed}$$

- In this limit, we effectively have a *continuous line* of non-susy CFT's parameterized by λ. At $\lambda = 0$ we reduce to the free fermionic vector model.
- All I said so far applies for fermion being in any representation, e.g. the adjoint.
- However, working with a vector fermion entails several simplifications so that certain exact results, and perhaps a complete large N solution, are possible.
- The analogous Chern-Simons bosonic vector model has been studied in parallel to our work in *Aharony et. al., 2011*.

Chern-Simons vector model

- I will discuss in particular two interesting results about the large N limit of this Chern-Simons vector model

1. The *exact* free energy of the theory on R^2 at finite temperature

$$F = -T \log Z_{R^2 \times S^1_\beta} = -\, \boxed{h(\lambda)}\, N V_2 T^3$$

   $h(\lambda)$ is a non-trivial function which we can compute *exactly* in λ.

2. At $N \to \infty$, for all λ, the theory admits an ∞-dimensional *higher spin symmetry*, i.e. there is an infinite tower of HS currents $J_s, s = 1, 2, 3, \ldots$ which are conserved at large N, so that

$$\Delta(J_s) = s + 1 + \mathcal{O}(\frac{1}{N}) \qquad \forall\, \lambda$$

Exact thermal free energy

- The Chern-Simons gauge field does not carry propagating degrees of freedom, so the theory is still essentially a vector model, and we expect it to be simpler than a typical large N gauge theory.
- However, the cubic self-interaction of the CS gauge field still makes perturbation theory complicated in general.
- Drastic simplifications can be achieved in a convenient gauge. We employ the "*light-cone gauge*"

$$A_- = 0 \qquad\qquad x^{\pm} = x^1 \pm i x^2$$

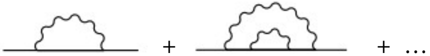
  Here x^1, x^2 are the Euclidean coordinates on R^2. The Euclidean time direction is x^3, which will be compactified on a circle of radius $\beta = 1/T$.
- In this gauge, the cubic self-interaction vanishes, and the large N free energy can be solved exactly.

Exact fermion propagator

- The basic ingredient we need to get the free energy is the exact fermion propagator

$$\langle \psi(p)^i \bar{\psi}(-p)_j \rangle = \delta_i^j \frac{1}{i p_\mu \gamma_\mu + \Sigma(p)}$$

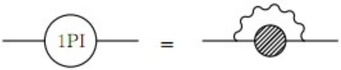
- $\Sigma(p)$ is the exact fermion self-energy. In the light-cone gauge and in the planar limit, it receives contributions only from 1PI rainbow diagrams



- Note that diagrams with matter loops do not contribute at leading order at large N, because the fermion is in the fundamental.

Exact fermion propagator

- It is not difficult to see that the sum of rainbow diagrams contributing to $\Sigma(p)$ satisfies the Schwinger-Dyson equation



$$\Sigma(p) = \frac{N}{2}\int \frac{d^3q}{(2\pi)^3}\left(\gamma^\mu \frac{1}{i\gamma^\alpha q_\alpha + \Sigma(q)}\gamma^\nu\right) G_{\mu\nu}(p-q)$$

- Here $G_{\mu\nu}(p)$ is the light-cone A_μ propagator: $G_{+3} = -G_{3+} = \frac{4\pi i}{kp^+}$.
- At finite temperature, we impose antiperiodic b.c. on the fermion, so

$$q^3 = \frac{2\pi}{\beta}(n+1/2), \qquad \int d^3q \to \int d^2q \sum_{\mathbb{Z}+1/2}$$

Exact fermion propagator

- The Schwinger-Dyson equation involves relatively simple loop integrals. However a suitable regularization is still needed. We employ the "dimensional reduction" scheme (shown to be consistent in CS-matter theories by *Chen, Semenoff, Wu '92* up to 2-loops).
- The self-energy which solves the Schwinger-Dyson equation takes the form

$$\Sigma(p) = f(\beta p_s) p_s + i\, g(\beta p_s) p^- \gamma^+ \qquad p_s^2 \equiv p_1^2 + p_2^2$$

where f, g satisfy a certain first order differential equation, whose solution is remarkably simple

$$f(y) = \frac{2\lambda}{y} \log\left(2\cosh\left[\frac{1}{2}\sqrt{c^2+y^2}\right]\right)$$
$$g(y) = \frac{c^2}{y^2} - f(y)^2.$$

Exact fermion propagator

- The solution depends on the constant $c = c(\lambda)$, which is the unique real solution to the transcendental equation

$$c = 2\lambda \log\left(2\cosh\frac{c}{2}\right)$$

- We see that this equation has no solution if $|\lambda| > 1$! So we conclude that the exact self-energy indicates that the CFT exist only for

$$0 \leq |\lambda| < 1$$

- In fact, we may give a simple explanation which has to do with the (regularization dependent) 1-loop shift of the CS level. In dim. red. there is no shift, however in YM regularization with bare CS level k_{YM} we have $k_{YM} \to k_{YM} + N$ so $\lambda = N/(k_{YM} + N) < 1$.

Exact thermal free energy

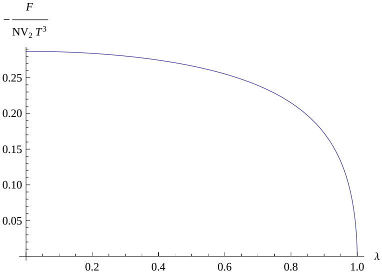
- Once we have the exact fermion self-energy Σ, one may show by path integral or diagrammatically that the free energy is given in terms of Σ by

$$F = NV_2 T \sum_n \int \frac{d^2 q}{(2\pi)^2} \mathrm{Tr} \left[\log \left[i\gamma^\mu q_\mu + \Sigma(q) \right] - \frac{1}{2} \Sigma(q) \left(\frac{1}{i\gamma^\mu q_\mu + \Sigma(q)} \right) \right]$$

- Performing the integral and sum, the final result is

$$F = -\frac{NV_2 T^3}{6\pi} \left[c^3 \frac{1-\lambda}{\lambda} + 6 \int_c^\infty dy \; y \log\left(1 + e^{-y}\right) \right] \equiv -NV_2 T^3 h(\lambda)$$

where $c = c(\lambda)$ is the constant introduced earlier.



$$h(\lambda) = \frac{3\zeta(3)}{4\pi} - \frac{2(\log 2)^3}{3\pi}\lambda^2 - \frac{(\log 2)^4}{2\pi}\lambda^4 + \ldots \qquad \lambda \ll 1$$

$$h(\lambda) \sim \frac{(1-\lambda)}{6\pi}\log^3(1-\lambda) + \ldots \qquad \lambda \to 1$$

- The function $h(\lambda)$ decreases monotonically from the free field value to zero at $\lambda = 1$. Extreme thinning of d.o.f. at "strong coupling". For comparison, in ABJM model we have $h(\lambda) \sim 1/\sqrt{\lambda}$ at $\lambda \to \infty$.

Higher spin symmetry at large N

- Recall that in the free theory ($\lambda = 0$), the spectrum of $U(N)$ invariant single trace primaries is

$$J_0 = \bar{\psi}_i \psi^i \,, \qquad J_s \sim \bar{\psi}_i \gamma_{(\mu_1} \partial_{\mu_2} \cdots \partial_{\mu_s)} \psi^i + \ldots$$

- In the interacting theory, these can be made gauge invariant by $\partial_\mu \to D_\mu$. The CS sector does not add any further single-trace primaries, because $(F_{\mu\nu})^i_j \sim \frac{1}{k} \bar{\psi}_j \gamma^\rho \psi^i \epsilon_{\mu\nu\rho}$ by e.o.m.
- In the free theory $\partial \cdot J_s = 0$, i.e. J_s are in short representations of the conformal algebra with $(\Delta, S) = (s+1, s)$.
- Turning on the interaction, we expect the currents not to be conserved any more and to acquire anomalous dimension $\Delta_s = s + 1 + \epsilon_s(\lambda, N)$.

- But for the currents to become non-conserved at $\lambda \neq 0$, we must have

$$\partial \cdot J_s \sim \lambda \mathcal{O}^{(s+2,s-1)}$$

In other words, there must be an operator in the $(s+2, s-1)$ representation with which J_s can combine to form a long representation.

- At $N = \infty$, single trace operators can only combine with other single trace operators. But there are no single-trace primaries in the spectrum with quantum numbers $(s+2, s-1)$!
- Therefore we conclude that at $N = \infty$, for all λ, the currents are still conserved, which implies

$$\Delta(J_s) = s + 1 + \mathcal{O}(\frac{1}{N}) \qquad \forall \ \lambda$$

- The vector nature of ψ is essential for this to work.

Higher spin symmetry at large N

- What happens is that, at finite N, J_s can (and does) combine with "double-trace" operators. The non-conservation equation takes the schematic form

$$\partial \cdot J_s \sim \frac{\lambda}{\sqrt{N}} \sum \partial^n J_{s_1} \partial^m J_{s_2}$$

This can be explicitly seen at leading order in λ by using the classical equations of motion. For example, for $s = 3$ one finds

$$\partial^\mu J^{(3)}_{\mu\nu_1\nu_2} = -\frac{16\pi\lambda}{5\sqrt{N}} \left[\eta_{\nu_1\nu_2} \left(\partial^\mu J^{(0)} \right) J^{(1)}_\mu - 3 \left(\partial_{(\nu_1} J^{(0)} \right) J^{(1)}_{\nu_2)} + 2J^{(0)} \partial_{(\nu_1} J^{(1)}_{\nu_2)} \right]$$

- The argument above implies that the HS currents do not have anomalous dimensions in the planar limit. But one can in fact argue that the scalar J_0 has protected dimension as well

$$\Delta(J_0) = 2 + \mathcal{O}(\frac{1}{N})$$

which we have checked perturbatively to two-loop order.

Comments on the holographic dual

- At $\lambda = 0$, we know that the theory should be dual to the Vasiliev's "type B" theory. So the holographic dual should be some deformation of such higher spin gravity theory.
- Turning on λ, we have seen that the spectrum of "single trace" primaries is

$$(\Delta, S) = (2 + \mathcal{O}(\frac{1}{N}), 0) + \sum_{s=1}^{\infty} (s + 1 + \mathcal{O}(\frac{1}{N}), s)$$

which implies that the dual bulk spectrum should contain classically massless higher spin fields and a $m^2 = -2$ scalar.
- The HS fields (and the scalar) can acquire mass via loop-corrections, but the bulk classical equations of motion should have exact higher spin gauge symmetry (to decouple longitudinal polarizations).
- Hence, the holographic dual should still be a higher spin gauge theory (with HS symmetry broken at quantum level), and it should break parity due to the boundary Chern-Simons term.

Comments on the holographic dual

- The only parity breaking higher spin gravity theory currently known is Vasiliev's theory specified by the general "interaction phase"

$$\Theta(X) = \theta_0 + \theta_2 X * X + \ldots$$

- A natural conjecture is then that our Chern-Simons vector model is dual to the parity breaking Vasiliev's theory with some specific choice

$$\theta_0(\lambda)\,, \qquad \theta_2(\lambda)\,, \qquad \ldots$$

  with the condition that $\theta_0(\lambda \to 0) = \pi/2$, $\theta_{2,4,\ldots}(\lambda \to 0) = 0$.
- We do not know a priori how to determine the phase as a function of λ. But we can in principle compute perturbatively correlators on both sides and compare.

Comments on the holographic dual

- From the calculation of 3-point functions in the bulk Vasiliev's theory with general phase, we have obtained

$$\langle JJJ \rangle = \cos^2\theta_0 \langle JJJ \rangle_B + \sin^2\theta_0 \langle JJJ \rangle_F$$

- The free scalar and free fermion structures are parity even, so they can arise in the CS vector model at even loop order only. By a 2-loop perturbative calculation, we find agreement with the above structure, and a (remarkably) simple result for the phase

$$\theta_0(\lambda) = \frac{\pi}{2}(1-\lambda) + \mathcal{O}(\lambda^3)$$

This is so simple that it is tempting to speculate it could be exact...Does the limit $\lambda \to 1$ have a description in terms of weakly coupled bosons?...

Comments on the holographic dual

- On the other hand, we do not find a parity odd contribution to $\langle JJJ \rangle$ from the bulk Vasiliev's side.
- At 1-loop order in the CS vector model, however, we do find a non-vanishing odd piece in $\langle J_1 J_1 T \rangle$, and $\langle TTT \rangle$, which would contradict the conjectured duality.
- Hopefully, there is a yet to be identified mistake or subtlety in the bulk calculation...Still work in progress!

Summary and conclusion

- Vasiliev's theory is a consistent non-linear HS gauge theory in AdS containing gravity, and we can apply the standard rules of holography to extract CFT correlation functions.
- We confirmed the Klebanov-Polyakov, Sezgin-Sundell conjectures at the level of 3-point functions.
- Chern-Simons vector models define lines of interacting non-susy CFT's with lagrangian description. They have higher spin symmetry at large N.
- We proposed a generalization of the KPSS conjecture which involves a parity breaking version of Vasiliev's HS gravity. Partial evidence, still work in progress.

Summary and conclusion

- The higher spin/vector models duality is interesting for several reasons
  - Explicit holographic dual of a free theory, but also potential exact dual of interacting CFT's such as Chern-Simons coupled vector models.
  - A "weak/weak" example of AdS/CFT where both sides are computable in the same regime.
  - Interesting toy model where to address theoretical questions about holography and quantum gravity.

- Many potential open problems
  - 4-point functions from Vasiliev's theory?
  - Study of exact solutions, e.g. black holes (*Didenko, Vasiliev '09, Iazeolla, Sundell '11)*
  - Free energy from the bulk HS theory? (Bulk action?)
  - Loop corrections in the bulk ($1/N$ expansion)?
  - Vasiliev theory in general AdS_d
  - ...