An action principle for Vasiliev's 4D equations

Nicolas Boulanger

Université de Mons, Belgium

11 April 2012, ESI

Based on 1102.2219[hep-th]

in collaboration with P. Sundell and a work to appear with P.S. and N. Colombo

Plan

The gauge principle [H. Weyl, 1929]

Classical Field Theory has witnessed a major achievement with Vasiliev's formulation of *fully nonlinear field equations* four higher-spin gauge fields in four space-time dimensions [M. A. Vasiliev, 1990 – 1992] and in D space-time dimensions [hep-th/0304049]. Some salient features are

- Manifest diffeomorphism invariance, no explicit reference to a metric
- Manifest Cartan integrability $\Rightarrow$ *gauge invariance* under infinite-dimensional HS algebra
- Formulation in terms of two infinite-dimensional unitarizable modules of $\mathfrak{so}(2, D-1)$: The *adjoint* and *twisted-adjoint* representations $\rightsquigarrow$ master 1-form and master zero-form, resp.

Unfolded equations and FDA

A free (graded commutative, associative) differential algebra $\mathfrak{R}$ is sets $\{X^\alpha\}$ of *a priori* independent variables that are locally-defined differential forms obeying first-order equations of motion whereby $\mathrm{d}X^\alpha$ are equated to algebraic functions of all the variables expressed entirely using the exterior algebra, *viz.*

$$\mathscr{R}^\alpha = \mathrm{d}X^\alpha + Q^\alpha(X) \approx 0 \,, \quad Q^\alpha(X) = \sum_n f^\alpha_{\beta_1\dots\beta_n} X^{\beta_1}\cdots X^{\beta_n} \,.$$

The nilpotency of d and the integrability condition $\mathrm{d}\mathscr{R}^\alpha \approx 0$ require

$$Q^\beta \, \frac{\partial^L Q^\alpha}{\partial X^\beta} \equiv 0 \quad .$$

For $X^\alpha_{[p_\alpha]}$ with $p_\alpha > 0$, gauge transformation preserving $\mathscr{R}^\alpha \approx 0$:

$$\delta_\epsilon X^\alpha = \mathrm{d}\epsilon^\alpha - \epsilon^\beta \frac{\partial^L}{\partial X^\beta} Q^\alpha \quad .$$

Why an action principle ?

At least three reasons why to search for action principles :

- At the classical level

  $\hookrightarrow$ explore non-perturbative aspects, different phases of the theory

- At the quantum level

  $\hookrightarrow$ try and find a consistent and suitable quantization scheme

- To shed a different light on Vasiliev's equations.

A prejudice : a QP-structure

We address this issue by using the fully non-linear and background-independent Vasiliev equations in four spacetime dimensions. These possess

- an algebraic structure that enables one to construct a *generalized Hamiltonian action* with nontrivial QP-structures in a manifold with boundary ;
- a geometric structure which allows to construct additional boundary deformations [$\longrightarrow$ Part II by Per Sundell].

Manifold : bulk with non-empty boundary

- Like for the Cattaneo–Felder model (nonlinear Poisson sigma-model), we introduce a bulk with non-empty boundary, and add extra momentum-like variables.
- Impose boundary conditions compatible with a *globally* well-defined action principle [the action should be invariant, the Lagrangian picks up a total derivative under general variation]
- Here we focus on the bulk part of the Hamitonian action. Various classically marginal deformations on submanifolds will be presented by Per Sundell in Part II.

Classical action principle (1)

Starting from $\{X^\alpha\}$ defined locally on B_ξ (where the base manifold $B_{\hat{p}} = \cup_\xi B_\xi$) satisfying some unfolded constraints with given Q-structure, $\hookrightarrow$ off-shell extensions based on sigma models with maps

$$\phi_\xi \ : \ T[1]B_\xi \ \to \ M_{\hat{p}} \ ,$$

between two $\mathbb{N}$-graded manifolds, from the parity-shifted tangent bundle $T[1]B_{\hat{p}}$ to a target space $M_{\hat{p}}$ that is a differential $\mathbb{N}$-graded symplectic manifold with two-form $\mathscr{O}$, Q-structure $\mathscr{Q}$ and Hamiltonian $\mathscr{H}$ with the following degrees :

$$\deg(\mathscr{O}) \ = \ \hat{p}+2 \ , \quad \deg(\mathscr{Q}) \ = \ 1 \ , \quad \deg(\mathscr{H}) \ = \ \hat{p}+1 \ .$$

Classical action principle (2)

$\hookrightarrow$ Classical action principle of Hamiltonian type :

$$S^{\rm cl}_{\rm bulk}[\phi] \;=\; \sum_{\xi}\int_{B_\xi}\mathscr{L}^{\rm cl}_{\xi} \;=\; \sum_{\xi}\int_{B_\xi}\mu\,\phi^{*}_{\xi}(\vartheta-\mathscr{H})\;,$$

where ϑ is the pre-symplectic form, defined locally on $M_{\hat{p}}$.

$\hookrightarrow$ Writing $\vartheta = {\rm d}Z^i\vartheta_i$, $\mathscr{O} = \frac{1}{2}\,{\rm d}Z^i{\rm d}Z^j\,\widetilde{\mathscr{O}}_{ij} = \frac{1}{2}\,{\rm d}Z^i\,\mathscr{O}_{ij}\,{\rm d}Z^j$ and defining

$$\{A,B\}^{[-\hat{p}]} = (-1)^{\hat{p}+(\hat{p}+i+1)A}\;\partial_i A\;\mathscr{P}^{ik}\;\partial_j B$$

where $\mathscr{P}^{ik}\mathscr{O}_{kj} = (-1)^{\hat{p}}\delta^i_j$, then ...

Classical action principle (3)

- ... the variation of the Lagrangian :

$$\delta \mathscr{L}^{\mathrm{cl}}_{\mathrm{bulk}} \;=\; \delta Z^i \mathscr{R}^j \widetilde{\mathscr{O}}_{ij} + \mathrm{d}\left(\delta Z^i \vartheta_i\right) \;,$$

where generalized curvatures and Hamiltonian vector field

$$\mathscr{R}^i \;=\; \mathrm{d}Z^i + \mathscr{Q}^i \;, \qquad \mathscr{Q}^i \;=\; (-1)^{\hat{p}+1} \mathscr{P}^{ij} \partial_j \mathscr{H} \;,$$
$$\overrightarrow{\mathscr{Q}} \;=\; \mathscr{Q}^i \vec{\partial}_i \;, \qquad \deg(\overrightarrow{\mathscr{Q}}) \;=\; 1 \;.$$

- Variational principle $\Longrightarrow$ $\mathscr{R}^i \approx 0$, whose Cartan integrability on shell requires $\overrightarrow{\mathscr{Q}}$ to be a Hamiltonian Q-structure

$$\mathscr{L}_{\overrightarrow{\mathscr{Q}}} \overrightarrow{\mathscr{Q}} \;\equiv\; 0 \quad \Leftrightarrow \quad \mathscr{Q}^j \partial_j \mathscr{Q}^i \;\equiv\; 0 \quad \Leftrightarrow \quad \partial_i \{\mathscr{H}, \mathscr{H}\}^{[-\hat{p}]} \;\equiv\; 0 \;.$$

Classical action principle (4)

Nilpotency of $\overrightarrow{\mathscr{Q}}$ with suitable boundary conditions on the fields and gauge parameters ensure invariance of the action under

$$\begin{aligned} \delta_\epsilon Z^i &= \mathrm{d}\epsilon^i - \epsilon^j \partial_j \mathscr{Q}^i + \tfrac{1}{2}\, \epsilon^k \mathscr{R}^l\, \partial_l \widetilde{\mathscr{O}}_{kj}\, \mathscr{P}^{ji}\ , \\ \delta_\epsilon \mathscr{L}^{\mathrm{cl}}_{\mathrm{bulk}} &= \mathrm{d}K_\epsilon\ , \qquad K_\epsilon \;=\; \epsilon^i \mathscr{R}^j \widetilde{\mathscr{O}}_{ij} + \delta_\epsilon Z^i \vartheta_i\ , \end{aligned}$$

Closure of gauge transformations :

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] Z^i \;=\; \delta_{\varepsilon_{12}} Z^i - \overrightarrow{\mathscr{R}} \varepsilon^i_{12}\ ,$$

where $\overrightarrow{\mathscr{R}} = \mathscr{R}^i \partial_i$ and

$$\varepsilon^i_{12} \;=\; -\tfrac{1}{2}\, [\overrightarrow{\varepsilon}_1, \overrightarrow{\varepsilon}_2]\, \mathscr{Q}^i\ .$$

Classical action principle (5)

- *Under certain extra assumptions* on ϑ and $\mathscr{H}$, the action can be defined globally by gluing together the locally defined fields and gauge parameters along chart boundaries using gauge transitions $\delta_t Z^i$ and $\delta_t \epsilon^i$ with parameters $\{t^i\} = t^{\xi}_{\xi'}$ defined on overlaps.

  Assumptions :

$$(i) \quad \delta_t K_\epsilon \;=\; 0 \,, \quad (ii) \quad \partial_j \partial_k \overrightarrow{t}\, \mathscr{Q}^i \;=\; 0 \,, \quad (iii) \quad K_t \;\equiv\; 0 \,.$$

- Assumption $(i) \Longrightarrow$ cancellation of contributions to $\delta_\epsilon S^{\text{cl}}_{\text{bulk}}$ from chart boundaries in the interior of B, s.t. the variational principle implies the BC

$$\vartheta_i|_{\partial B} \;=\; 0 \,.$$

Classical action principle (6)

- Assumption (ii) ensures compatibility between gauge transformations and gauge transitions in the sense that performing a transition transformation on fields and gauge parameters between two adjacent charts and moving along the gauge orbit are two operations that should commute.
- Assumption (iii) selects the subalgebra of Cartan transformations that preserve the Lagrangian density, *i.e.* selects the transitions.
- Assuming there are no constants of total degree $\hat{p}+2$ on $M_{\hat{p}}$, the condition $\partial_i\{\mathscr{H},\mathscr{H}\}^{[-\hat{p}]}\equiv 0$ is equivalent to the *structure equation*

$$\{\mathscr{H},\mathscr{H}\}^{[-\hat{p}]} \quad\equiv\quad 0 \quad\Leftrightarrow\quad (-1)^{i(\hat{p}+1)}\,\partial_i\mathscr{H}\,\mathscr{P}^{ij}\partial_j\mathscr{H} \;\equiv\; 0\,.$$

Brief review of Vasiliev's 4D equations (1)

The master fields are locally-defined (chart index ξ) **operators**

$$O_\xi(X^M_\xi, \mathrm{d}X^M_\xi; Z^{\underline{\alpha}}, \mathrm{d}Z^{\underline{\alpha}}; Y^{\underline{\alpha}}; K) \ ,$$

where

$$[Y^{\underline{\alpha}}, Y^{\underline{\beta}}] \ = \ 2iC^{\underline{\alpha\beta}} \ , \quad [Z^{\underline{\alpha}}, Z^{\underline{\beta}}] \ = \ -2iC^{\underline{\alpha\beta}} \ ,$$

with charge conjugation matrix $C^{\alpha\beta} = \epsilon^{\alpha\beta}$, $C^{\dot\alpha\dot\beta} = \epsilon^{\dot\alpha\dot\beta}$ and where $K = (k, \bar{k})$, are two outer Kleinian operators.

The operators are represented by **symbols** $f[O_\xi]$ obtained by going to **specific bases** for the operator algebra $\rightsquigarrow$ **ordering prescriptions**.

Brief review of Vasiliev's 4D equations (2)

One may think of the symbols as functions $f(X, Z; \mathrm{d}X, \mathrm{d}Z; Y)$ on a **correspondence space** $\mathfrak{C}$

$$\mathfrak{C} \; = \; \bigcup_{\xi} \mathfrak{C}_\xi \; , \qquad \mathfrak{C}_\xi \; = \; \mathfrak{B}_\xi \times \mathfrak{Y} \; , \qquad \mathfrak{B}_\xi \; = \; \mathfrak{M}_\xi \times \mathfrak{Z}$$

equipped with a suitable `associative` star-product operation $\star$ which reproduces, in the space of symbols, the composition rule for operators.

$\looparrowright$ The exterior derivative on $\mathfrak{B}$ is given by

$$\mathrm{d} \; = \; \mathrm{d}X^M \partial_M + q \; , \qquad q \; = \; \mathrm{d}Z^{\underline{\alpha}} \, \partial_{\underline{\alpha}} \; .$$

Brief review of Vasiliev's 4D equations (3)

The master fields of the *minimal bosonic model* are an adjoint one-form

$$A = W + V \ ,$$

$$W = \mathrm{d}X^M \, W_M(X,Z;Y) \ , \qquad V = \mathrm{d}Z^{\underline{\alpha}} \, V_{\underline{\alpha}}(X,Z;Y) \ ,$$

and a twisted-adjoint zero-form

$$\Phi = \Phi(X,Z;Y) \ .$$

Generically, start with locally-defined differential forms of *total degree* p

$$f = \sum_{p=0}^{\infty} f_{[p]}(X^M, \mathrm{d}X^M; Z^{\underline{\alpha}}, \mathrm{d}Z^{\underline{\alpha}}; Y^{\underline{\alpha}}; k, \bar{k}) \ ,$$

$$f_{[p]}(\lambda \, \mathrm{d}X^M; \lambda \, \mathrm{d}Z^{\underline{\alpha}}) = \lambda^p \, f_{[p]}(\mathrm{d}X^M; dZ^{\underline{\alpha}}) \ , \quad \lambda \in \mathbb{C} \ .$$

Brief review of Vasiliev's 4D equations (4)

The X^M's are commuting coordinates, while $(Y^{\underline{\alpha}}, Z^{\underline{\alpha}}) = (y^\alpha, \bar{y}^{\dot{\alpha}}; z^\alpha, \bar{z}^{\dot{\alpha}})$ are non-commutative *twistor-space* coordinates and $k\,,\ \bar{k}$ are outer Kleinians :

$$k \star f \ = \ \pi(f) \star k \ , \quad \bar{k} \star f \ = \ \bar{\pi}(f) \star \bar{k} \ , \quad k \star k \ = \ 1 \ = \ \bar{k} \star \bar{k} \ ,$$

with automorphisms π and $\bar{\pi}$ defined by $\pi \mathrm{d} = \mathrm{d}\pi\,,\ \bar{\pi}\mathrm{d} = \mathrm{d}\bar{\pi}$ and

$$\pi[f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}})] \ = \ f(-z^\alpha, \bar{z}^{\dot{\alpha}}; -y^\alpha, \bar{y}^{\dot{\alpha}}) \ ,$$
$$\bar{\pi}[f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}})] \ = \ f(z^\alpha, -\bar{z}^{\dot{\alpha}}; y^\alpha, -\bar{y}^{\dot{\alpha}}) \ .$$

Bosonic and irreducibility projections : $\quad \pi\bar{\pi}(f) = f = P_+ \star f \ ,$
$P_+ = \frac{1}{2}(1 + k \star \bar{k})\,,$

$$\hookrightarrow \ f = \ \left[f^{(+)}(X, \mathrm{d}X; Z, \mathrm{d}Z; Y) + f^{(-)}(X, \mathrm{d}X; Z, \mathrm{d}Z; Y) \star \frac{(k + \bar{k})}{2} \right] \star P_+ \ .$$

Brief review of Vasiliev's 4D equations (5)

- **Bosonic projection** : removes component fields $\rightsquigarrow$ spacetime spinors.
- **Irreducible *minimal* bosonic models : by imposing** reality conditions and discrete symmetries that remove all **odd** spins.

$\hookrightarrow$ $\dagger$ and anti-automorphism τ defined by $\mathrm{d}[(\cdot)^\dagger] = [\mathrm{d}(\cdot)]^\dagger\,, \quad \mathrm{d}\,\tau = \tau\,\mathrm{d}\,,$

$$
\begin{aligned}
[f(z^\alpha,\bar z^{\dot\alpha};y^\alpha,\bar y^{\dot\alpha};k,\bar k)]^\dagger &= \bar f(\bar z^{\dot\alpha},z^\alpha;\bar y^{\dot\alpha},y^\alpha;\bar k,k)\ ,\\
\tau[f(z^\alpha,\bar z^{\dot\alpha};y^\alpha,\bar y^{\dot\alpha};k,\bar k)] &= f(-iz^\alpha,-i\bar z^{\dot\alpha};iy^\alpha,i\bar y^{\dot\alpha};k,\bar k)\ ,\\
[f_{[p]}\star f'_{[p']}]^\dagger &= (-1)^{pp'}(f'_{[p']})^\dagger\star(f_{[p]})^\dagger\ ,\\
\tau(f_{[p]}\star f'_{[p']}) &= (-1)^{pp'}\tau(f'_{[p']})\star\tau(f_{[p]})\ .
\end{aligned}
$$

Brief review of Vasiliev's 4D equations (6)

Back to Vasiliev's A and Φ, the minimal models are imposed by the following projection and reality conditions :

$$\tau(A,\Phi) \;=\; (-A,\pi(\Phi)) \;, \qquad (A,\Phi)^\dagger \;=\; (-A,\pi(\Phi)) \;.$$

Full equations of motion of the minimal bosonic model with fixed interaction ambiguity : $F+\Phi\star J=0$, with two-form J defined globally on correspondence space, obeying $\tau(J)=-J=J^\dagger$ and

$$\mathrm{d}J \;=\; 0 \;, \quad [f,J]^\pi_\star \;:=\; f\star J - J\star\pi(f) \;=\; 0 \quad \forall\, f \quad \text{s.t.} \quad \pi\bar\pi(f)=f \;. \quad (1)$$

In the minimal model,

$$J \;=\; -\frac{i}{4}(b\,dz^2\;\kappa+\bar b\,d\bar z^2\;\bar\kappa) \;,$$

Brief review of Vasiliev's 4D equations (7)

... where the chiral inner Kleinians

$$\kappa \;=\; \exp(i y^\alpha z_\alpha)\;, \qquad \bar\kappa \;=\; \kappa^\dagger = \exp(-i \bar y^{\dot\alpha} \bar z_{\dot\alpha})\;.$$

By making use of field redefinitions $\Phi \to \lambda\Phi$ with $\lambda \in \mathbb{R}\,,\ \lambda \neq 0\,,$ the complex parameter b in J can be taken to obey

$$|b| \;=\; 1\;, \qquad \arg(b) \;\in\; [0,\pi]\;.$$

The phase breaks parity P $\quad [P\,\mathrm{d} = \mathrm{d}\,P]$

$$P\left[f(X^M; z^\alpha, \bar z^{\dot\alpha}; y^\alpha, \bar y^{\dot\alpha}; k, \bar k)\right] \;=\; (Pf)(X^M; -\bar z^{\dot\alpha}, -z^\alpha; \bar y^{\dot\alpha}, y^\alpha; \bar k, k)\;,$$

except in the following two cases :

$$\text{Type-A model (parity-even physical scalar)} \quad : \quad b = 1\;,$$

$$\text{Type-B model (parity-odd physical scalar)} \quad : \quad b = i\;.$$

Brief review of Vasiliev's 4D equations (8)

[The integrability of $F + \Phi \star J = 0$ implies that $D\Phi \star J = 0$, that is, $D\Phi = 0$, where the twisted-adjoint covariant derivative $D\Phi = d\Phi + A \star \Phi - \Phi \star \pi(A)$. This constraints is integrable since $D^2\Phi \;=\; F \star \Phi - F \star \pi(\Phi) \;=\; -\Phi \star J \star \Phi + \Phi \star \pi(\Phi) \star J$ gives zero, using the constraint on F and (1).]

$\hookrightarrow$ Summary : minimal higher-spin gravity given by

$$F + \Phi \star J \;=\; 0 \;, \qquad D\,\Phi \;=\; 0 \;, \qquad \mathrm{d}J = 0 \;,$$

$$F \;:=\; \mathrm{d}A + A \star A \;, \qquad D\Phi \;:=\; \mathrm{d}\Phi + [A, \Phi]_\pi \;,$$

$$\tau(A, \Phi) \;=\; (-A, \pi(\Phi)) \;, \qquad (A, \Phi)^\dagger \;=\; (-A, \pi(\Phi)) \;,$$

$$\hookrightarrow [A, J]_\pi = 0 = [\Phi, J]_\pi \;.$$

Brief review of Vasiliev's 4D equations (9)

$\hookrightarrow$ Integrability implies invariance under Cartan gauge transformations

$$\delta_\epsilon A \;= D\epsilon \;, \qquad \delta_\epsilon \Phi \;=\; -[\epsilon, \Phi]_\pi \;,$$

for zero-form gauge parameters $\epsilon(X, Z; Y)$ obeying the same kinematic constraints as the master one-form, *i.e.* $\tau(\epsilon) = -\epsilon$ and $(\epsilon)^\dagger = -\epsilon\,.$

$\hookrightarrow$ The closure of the gauge transformations reads

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \;=\; \delta_{\epsilon_{12}} \;, \qquad \epsilon_{12} \;=\; [\epsilon_1, \epsilon_2]_\star \;,$$

defining the algebra $\mathfrak{hs}(4)\,.$

Hamiltonian action principle ; chiral trace

$\hookrightarrow$ Integration over $\mathfrak{C}$ of a globally-defined $(\hat{p}+1)$-form $\mathscr{L}$:

$$\int_{\mathfrak{C}} \mathscr{L} \;=\; \sum_{\xi} \int_{M_\xi} \mathrm{Tr}\,[f_{\mathscr{L}}] \;\;,$$

where $f_{\mathscr{L}}$ denotes a symbol of $\mathscr{L}$ and the chiral trace operation is defined by

$$\mathrm{Tr}\,[f] \;=\; \sum_{m} \int_{\mathfrak{Z}\times\mathfrak{Y}} \frac{d^2y d^2\bar{y}}{(2\pi)^2} \frac{f_{[m;2,2]}|_{k=0=\bar{k}}}{(2\pi)^2} \;\;, \qquad (2)$$

using $f_{[p]} = \sum_{\substack{m\,+\,q\,+\,\bar{q}\,=\,p \\ q,\,\bar{q}\,\leqslant\,2}} f_{[m;q,\bar{q}]}$ with

$$f_{[m;q,\bar{q}]}(\lambda\, dX^M; \mu\, dz^\alpha, \bar{\mu}\, d\bar{z}^{\dot{\alpha}}) \;=\; \lambda^m\, \mu^q\, \bar{\mu}^{\bar{q}}\, f_{[m;q,\bar{q}]}(dX^M; dz^\alpha, d\bar{z}^{\dot{\alpha}}) \;. \qquad (3)$$

One integrates over $\{y^\alpha, z^\alpha\}$ and $\{\bar{y}^{\dot{\alpha}}, \bar{z}^{\dot{\alpha}}\}$ viewed as real, independent variables.

Action principle ; Graded cyclic trace

This choice implies

$$\mathrm{Tr}\,[\pi(f)] \;=\; \mathrm{Tr}\,[\bar{\pi}(f)] \;=\; \mathrm{Tr}\,[f] \;\;,$$

which in its turn implies graded cyclicity,

$$\mathrm{Tr}\left[f_{[p]} \star f'_{[p']}\right] \;=\; (-1)^{pp'}\;\mathrm{Tr}\left[f'_{[p']} \star f_{[p]}\right] \;\;,$$

Furthermore

$$(\mathrm{Tr}\,[f])^{\dagger} \;=\; \mathrm{Tr}\left[(f)^{\dagger}\right] \;\;, \quad \mathrm{Tr}\,[P(f)] \;=\; \mathrm{Tr}\,[f] \;\;, \quad \mathrm{Tr}\,[\pi_k(f)] \;=\; \mathrm{Tr}\,[f] \;\;, \quad \text{where}$$

$$\pi_k \;:\; (k,\bar{k}) \;\mapsto\; (-k,-\bar{k}) \;,$$

$$P[f(X^M;z^{\alpha},\bar{z}^{\dot{\alpha}};y^{\alpha},\bar{y}^{\dot{\alpha}};k,\bar{k})] \;=\; (Pf)(X^M;-\bar{z}^{\dot{\alpha}},-z^{\alpha};\bar{y}^{\dot{\alpha}},y^{\alpha};\bar{k},k) \;.$$

[where Pf is expanded in terms of parity-reversed component fields.]

Odd-dimensional bulk ($\hat{p} \in 2\mathbb{N}$)

$\hookrightarrow$ Finally, we assume that, off shell : $\mathrm{Tr}\,[\tau(f)] \;=\; \mathrm{Tr}\,[f]$, and that the integration over $\mathfrak{C}$ is non-degenerate : If $\mathrm{Tr}\,[f \star g] = 0$ for all f, then $g = 0$.

In the case of an odd-dimensional base manifold of dimension $\hat{p} + 1 = 2n + 5$ with $n \in \{0, 1, 2, \ldots\}$ such that $\dim(M) = 2n + 1$, we propose the bulk action

$$S^{\mathrm{cl}}_{\mathrm{bulk}}[\{A, B, U, V\}_\xi] \;\;=\;\; \sum_\xi \int_{M_\xi} \mathrm{Tr}\left[U \star DB + V \star \Big(F + \mathscr{G}(B, U; J^I, J^{\bar{I}}, J^{I\bar{I}})\Big)\right]$$

with interaction freedom $\mathscr{G}$ and locally-defined master fields $(m = n + 2)$

$$\begin{aligned} A &= A_{[1]} + A_{[3]} + \cdots + A_{[2m-1]} \,, \qquad & B &= B_{[0]} + B_{[2]} + \cdots + B_{[2m-2]} \,, \\ U &= U_{[2]} + U_{[4]} + \cdots + U_{[2m]} \,, \qquad & V &= V_{[1]} + V_{[3]} + \cdots + V_{[2m-1]} \,. \end{aligned}$$

Why such an extension?

- Because we want a P-structure and only wedge products in the Lagrangian, (take $n=2$ here) $U_{[8]}$ and $V_{[7]}$ are not sufficient : $U_{[8]} \star V_{[7]}$ is not of total degree $9 = 4+1+4$.
- $\mathscr{G}$ must be constrained in order for the action to be gauge invariant and in order to avoid systems that are trivial. We take

$$\begin{aligned}
\mathscr{G} &= \mathscr{F}(B; J^I, J^{\bar{I}}, J^{I\bar{I}}) + \widetilde{\mathscr{F}}(U; J^I, J^{\bar{I}}, J^{I\bar{I}}) \quad , \\
\mathscr{F} &= \mathscr{F}_I(B) \star J^I_{[2]} + \mathscr{F}_{\bar{I}}(B) \star J^{\bar{I}}_{[2]} + \mathscr{F}_{I\bar{I}}(B) \star J^{I\bar{I}}_{[4]} \quad , \\
\widetilde{\mathscr{F}} &= \widetilde{\mathscr{F}}_I(U) \star J^I_{[2]} + \widetilde{\mathscr{F}}_{\bar{I}}(U) \star J^{\bar{I}}_{[2]} + \widetilde{\mathscr{F}}_{I\bar{I}}(U) \star J^{I\bar{I}}_{[4]} \quad ,
\end{aligned}$$

where the central and closed elements

$$(J^I_{[2]})_{I=1,2} = -\tfrac{i}{4}(1\,,\,k\kappa) \star P_+ \star d^2 z\ ,\ (J^{\bar{I}}_{[2]})_{\bar{I}=\bar{1},\bar{2}} = -\tfrac{i}{4}(1\,,\,\bar{k}\bar{\kappa}) \star P_+ \star d^2 \bar{z}$$

$$J^{I\bar{I}}_{[4]} = 4\,i\,J^I_{[2]} J^{\bar{I}}_{[2]} \quad ,$$

Interaction freedom

Denoting $Z^i = (A, B, U, V)$, the general variation of the action defines generalized curvatures $\mathscr{R}^i$ as follows :

$$\delta S \ = \ \sum_\xi \int_{M_\xi} \mathrm{Tr}\left[\mathscr{R}^i \star \delta Z^j \mathscr{O}_{ij}\right] + \sum_\xi \int_{\partial M_\xi} \mathrm{Tr}\left[U \star \delta B - V \star \delta A\right] \ ,$$

where one thus has

$$\mathscr{R}^A \ = \ F + \mathscr{F} + \widetilde{\mathscr{F}} \ , \qquad \mathscr{R}^B \ = \ DB + (V\partial_U) \star \widetilde{\mathscr{F}} \ ,$$

$$\mathscr{R}^U \ = \ DU - (V\partial_B) \star \mathscr{F} \ , \qquad \mathscr{R}^V \ = \ DV + [B, U]_\star \ ,$$

with $\mathscr{O}_{ij}$ being a constant non-degenerate matrix (defining a symplectic form of degree $\hat{p} + 2$ on the $\mathbb{N}$-graded target space of the bulk theory).

Obstruction to Cartan integrability ?

Generically there are obstructions to Cartan integrability of the unfolded equations of motion $\mathscr{R}^i \approx 0$. These obstructions vanish identically (without further algebraic constraints on Z^i) in at least the following two cases :

$$\begin{aligned} \text{bilinear } Q\text{-structure} \quad &: \quad \mathscr{F} \;=\; B \star J\;, \qquad J \;=\; J_{[2]} + J_{[4]}\;, \\ \text{bilinear } P\text{-structure} \quad &: \quad \widetilde{\mathscr{F}} \;=\; U \star J'\;, \qquad J' \;=\; J'_{[2]} + J'_{[4]}\;. \end{aligned}$$

where $B \star J_{[2]} = B \star (b_I\, J^I_{[2]} + b_{\bar I}\, J^{\bar I}_{[2]})\,,\;\; B \star J_{[4]} = B \star (c_{I\bar I}\, J^{I\bar I}_{[4]})$, idem J'.

Consistency

Recall that if $\mathscr{R}^i = dZ^i + \mathscr{Q}^i(Z^j)$ defines a set of generalized curvatures, then one has the following three equivalent statements :

(I) $\mathscr{R}^i$ obey a set of generalized Bianchi identities $d\mathscr{R}^i - (\mathscr{R}^j \partial_j) \star \mathscr{Q}^i \equiv 0\,;$

(II) $\mathscr{R}^i$ transform into each other under Cartan gauge transformations $\delta_\varepsilon Z^i = d\varepsilon^i - (\varepsilon^j \partial_j) \star \mathscr{Q}^i$; and

(III) the quantity $\overrightarrow{\mathscr{Q}} := \mathscr{Q}^i \partial_i$ is a Q-structure, *i.e.* a nilpotent $\star$-vector field of degree one in target space, *viz.* $\overrightarrow{\mathscr{Q}} \star \mathscr{Q}^i \equiv 0\,.$

Furthermore, in the case of differential algebras on commutative base manifolds, one can show that if $\mathscr{R}^i$ are defined via a variational principle as above (with constant $\mathscr{O}_{ij}$), then the action S remains invariant under $\delta_\varepsilon Z^i\,.$

Cartan gauge transformations

In the two Cartan integrable cases at hand, one thus has the on-shell Cartan gauge transformations

$$
\begin{aligned}
\delta_{\epsilon,\eta} A &= D\epsilon^{A} - (\epsilon^{B}\partial_B)\star \mathscr{F} - (\eta^{U}\partial_U)\star \widetilde{\mathscr{F}}\ , \\
\delta_{\epsilon,\eta} B &= D\epsilon^{B} - [\epsilon^{A}, B]_\star - (\eta^{V}\partial_U)\star \widetilde{\mathscr{F}} - (\eta^{U}\partial_U)\star (V\partial_U)\star \widetilde{\mathscr{F}}\ , \\
\delta_{\epsilon,\eta} U &= D\eta^{U} - [\epsilon^{A}, U]_\star + (\eta^{V}\partial_B)\star \mathscr{F} + (\epsilon^{B}\partial_B)\star (V\partial_B)\star \mathscr{F}\ , \\
\delta_{\epsilon,\eta} V &= D\eta^{V} - [\epsilon^{A}, V]_\star - [\epsilon^{B}, U]_\star + [\eta^{U}, B]_\star\ .
\end{aligned}
$$

These transformations remain symmetries off shell, although we are in the context of non-graded commutative (but still associative) target-space (here viewed as base) manifold.

Cartan gauge algebra

$\longmapsto$ More precisely, the $(\epsilon^A;\epsilon^B)$-symmetries leave the Lagrangian invariant while the (η^U,η^V)-symmetries transform the Lagrangian into a nontrivial total derivative, *viz.*

$$\delta_{\epsilon,\eta}\mathscr{L} \;\equiv\; \mathrm{d}\left(Tr\left[\eta^U\star\mathscr{K}_U+\eta^V\star\mathscr{K}_V\right]\right)\;,$$

for $(\mathscr{K}_U,\mathscr{K}_V)$ that are not identically zero. It follows that the Cartan gauge algebra $\mathfrak{g}$ is of the form

$$\mathfrak{g}\cong\mathfrak{g}_1 \Subset\!\!\!\!+\, \mathfrak{g}_2$$

with $\mathfrak{g}_1\cong\mathrm{span}\{\epsilon^A,\epsilon^B\}$ and $\mathfrak{g}_2\cong\mathrm{span}\{\eta^U,\eta^V\}$, as one can verify explicitly.

$\longmapsto$ In order for the variational principle to be globally well-defined, one has (like in Cattaneo–Felder-like analysis) to impose the following :

$$(U,V)|_{\partial M}\;=\;0\;.$$

PERTURBATIVE EQUIVALENCE

The duality-extended (A, B)-system is perturbatively equivalent to Vasiliev's original $(A_{[1]}, B_{[0]})$-system :

(I) both systems share the same Weyl zero-form $B_{[0]}$; this master field contains the initial data associated to the Weyl curvature tensors, which contain one-particle states and other local deformations of the system.

(II) the master fields with positive form degree (including $A_{[1]}$) bring in gauge functions. In topologically (softly) broken phases, the boundary values of gauge functions associated with topologically broken gauge symmetries may contribute to observables. Thus in the unbroken phase (where no gauge functions are observable) the original and duality-extended systems share the same observable gauge functions.

Conclusions and outlook

Action principle for Vasiliev's systems, which admits consistent truncation to minimal models.

The duality-extended (A, B)-system is perturbatively equivalent to Vasiliev's original $(A_{[1]}, B_{[0]})$-system

Starting point for quantization and addition of boundary deformations $\rightarrow$ next talk by Per S.