Carrollian limit of quadratic gravity

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Overview

1. Quadratic gravity
2. PUL parameterization and Carrollian expansion
3. The Carrollian limit of quadratic gravity
4. Outlook

Quadratic gravity

The action in 4 dimensions is

$$S = \int d^4xc^3\sqrt{-g}[\frac{1}{16\pi G}R - \alpha R^{\mu\nu}R_{\mu\nu} + \beta R^2],$$

- Improves the UV behaviour of GR (Renormalizable or even super renormalizable).
- Introduces additional degrees of freedom including ghosts (can be tachyonic).
- Tachyon removing conditions

$$\alpha \leq 0,$$

$$\alpha - 3\beta \geq 0.$$

- Has black hole solutions (Schwarzchild-Bach) not found in GR.

PUL parameterization

- Convenient to taking the Carrollian (ultra local) limit.
- The PUL parameterization (Hansen et al., 2022) of the metric (mostly positive) is given by

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}, \quad g^{\mu\nu} = -\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu},$$

  where $\Pi_{\mu\nu}$ is the induced metric on space slices.
- Expand the quantities in powers of c^2

$$V^\mu = v^\mu + c^2 M^\mu + O(c^4), \quad T_\mu = \tau_\mu + c^2 N_\mu + O(c^4)$$

$$\Pi^{\mu\nu} = h^{\mu\nu} + c^2 \Phi^{\mu\nu} + O(c^4), \quad \Pi_{\mu\nu} = h_{\mu\nu} + c^2 \Phi_{\mu\nu} + O(c^4)$$

  where $v^\mu, M^\mu, \tau_\mu, N_\mu, \Phi^{\mu\nu}, \Phi_{\mu\nu}$ are the fields used in the theory.
- Derive other quantities accordingly.

Highlights of the Carrollian limit of GR

The action to the leading order is

$$S = \frac{c^2}{16\pi G}\int [K^2 - K_{\mu\nu}K^{\mu\nu}]Ed^4x,$$

where E is the determinant of the vielbein, and $K_{\mu\nu}$ is the extrinsic curvature of spatial surfaces. The equations of motion are

$$K^2 - K^{\mu\nu}K_{\mu\nu} = 0,$$

$$h^{\nu\alpha}\nabla_\alpha[K_{\mu\nu} - Kh_{\mu\nu}] = 0.$$

$$\mathcal{L}_v K_{\mu\nu} = -2K^\alpha_\mu K_{\nu\alpha} + KK_{\mu\nu},$$

- Unlike the Galilean limit, the leading order equations have non trivial field equations.

The Carrollian limit of quadratic gravity

The action of quadratic gravity is

$$S = \int d^4xc^3\sqrt{-g}[\frac{1}{16\pi G}R - \alpha R^{\mu\nu}R_{\mu\nu} + \beta R^2],$$

where α and β are parameters.

The first order of the PUL parameterizations of the quantities in the action are

$$\begin{aligned} R^{\mu\nu}R_{\mu\nu} = \frac{1}{c^4}(& h^{\nu\alpha}h^{\lambda\beta}\nabla_\mu(v^\mu K_{\alpha\beta})\nabla_\rho(v^\rho K_{\nu\lambda}) - 2K^{\alpha\beta}K\nabla_\mu(v^\mu K_{\alpha\beta}) \\ & + K^{\lambda\nu}K_{\lambda\nu}K^2 - v^\lambda v^\nu \nabla_\mu(K)\nabla_\nu(K) + 2K_{\alpha\beta}K^{\alpha\beta}v^\nu\nabla_\nu K \\ & - (K^{\mu\nu}K_{\mu\nu})^2) \end{aligned}$$

The Carrollian limit of quadratic gravity

$$
\begin{aligned}
R^2 = \frac{1}{c^4}[&h^{\lambda\nu}h^{\sigma\rho}\nabla_\mu(v^\mu K_{\lambda\nu})\nabla_\alpha(v^\alpha K_{\sigma\rho}) - 2h^{\lambda\nu}\nabla_\mu(v^\mu K_{\lambda\nu})K^2 \\
&+ 2h^{\lambda\nu}\nabla_\mu(v^\mu K_{\lambda\nu})K^{\alpha\beta}K_{\alpha\beta} + 2h^{\lambda\nu}\nabla_\mu(v^\mu K_{\lambda\nu})v^\alpha\nabla_\alpha K \\
&+ K^4 - 2K^2K^{\mu\nu}K_{\mu\nu} - 2K^2v^\mu\nabla_\mu(K)(K^{\mu\nu}K_{\mu\nu})^2 \\
&+ 2K^{\mu\nu}K_{\mu\nu}v^\alpha\nabla_\alpha K + v^\mu\nabla_\mu K v^\nu\nabla_\nu K]
\end{aligned}
$$

- R^2 and $R_{\mu\nu}R^{\mu\nu} = O(c^{-4})$ while $R = O(c^{-2})$.
- The leading order would not involve R except if α and β depend on c.

The dependencies of α and β on c

Denoting the theory resulting from choosing $\alpha = c^n\alpha'$ and $\beta = c^m\beta'$ by (n, m), we get the following

Carrollian theories from quadratic gravity		
Theory	Action contributing to the LO	Type of modification to the Carrollian limit of GR
(0,0)	$S = c^3 \int \left[-\alpha R^{\mu\nu}R_{\mu\nu} + \beta R^2 \right] \sqrt{-g} d^4x$	*Not a modification of GR*
(0,2)	$S = c^3 \int -\alpha R^{\mu\nu}R_{\mu\nu}\sqrt{-g} d^4x$	*Not a modification of GR*
(2,0)	$S = c^3 \int \beta R^2 \sqrt{-g} d^4x$	*Not a modification of GR*
(2,2)	$S = c^3 \int \left[R - \alpha R^{\mu\nu}R_{\mu\nu} + \beta R^2 \right] \sqrt{-g} d^4x$	*Modifies GR to the LO*
(2,4)	$S = c^3 \int \left[R - \alpha R^{\mu\nu}R_{\mu\nu} \right] \sqrt{-g} d^4x$	*Modifies GR to the LO with* $R^{\mu\nu}R_{\mu\nu}$ *terms and the NLO by* R^2 *terms*
(4,2)	$S = c^3 \int \left[R + \beta R^2 \right] \sqrt{-g} d^4x$	*Modifies GR to the LO with* R^2 *terms and the NLO by* $R^{\mu\nu}R_{\mu\nu}$ *terms*
(4,4)	$S = c^3 \int R\sqrt{-g} d^4x$	*Modifies GR in the NLO*

The dependencies of α and β on c

- (0,0)
- (0,2)
- (2,0)

} Pure quadratic gravity

- (2,2)
- (4,4)

} Stelle's gravity. Have ghosts.
Tachyon removing conditions: $\alpha' \leq 0$, $\alpha' - 3\beta' \geq 0$

- (2,4)
- (4,2)

} $R + R^2$ theories. No ghosts.
Tachyon removing conditions: $\alpha' = 0$, $\beta' \leq 0$

We focus on (2,4) and (4,2).

(2,4) theory in the electric limit

Same as GR. The action to the leading order is

$$S = \frac{c^2}{16\pi G} \int [K^2 - K_{\mu\nu} K^{\mu\nu}] E d^4 x.$$

The equations of motion are

$$K^2 - K^{\mu\nu} K_{\mu\nu} = 0,$$

$$h^{\nu\alpha} \nabla_\alpha [K_{\mu\nu} - K h_{\mu\nu}] = 0.$$

$$\mathcal{L}_v K_{\mu\nu} = -2 K_\mu^\alpha K_{\nu\alpha} + K K_{\mu\nu}.$$

(2,4) theory in the magnetic limit

The action is

$$S = -\int d^4xe\Big[-\overset{c}{R} + \beta'\big[(K^2 - K_{\mu\nu}K^{\mu\nu})(K^2 - K_{\mu\nu}K^{\mu\nu} + 4\pounds_{\boldsymbol{v}}K) - 4(\pounds_{\boldsymbol{v}}K)^2\big],$$

where $\overset{c}{R}$ is the Ricci scalar for the compatible connection chosen.

- The action is subjugated to the constraints

$$K^2 - K^{\mu\nu}K_{\mu\nu} = 0$$

$$\pounds_v K = K^2$$

- The LO equations must be included at the equations for the magnetic theory.

(4,2) theory in the electric limit

The action is

$$S = \int d^4xe[(K^2 - K_{\mu\nu}K^{\mu\nu})(1+\beta'[K^2 - K^{\mu\nu}K_{\mu\nu} + 4\pounds_{\boldsymbol{v}}K]) - \beta'(\pounds_{\boldsymbol{v}}K)^2].$$

Varying the action we get the constraints

$$(K^2 - K_{\mu\nu}K^{\mu\nu})(1 + \beta'[K^2 - K^{\mu\nu}K_{\mu\nu} + 4\pounds_{\boldsymbol{v}}K]) - \beta'(\pounds_{\boldsymbol{v}}K)^2 = 0,$$

$$h^{\mu\rho}\nabla_\mu(K_{\rho\nu} - Kh_{\rho\nu}) + 2\beta'[K_{\rho\nu}(-3(K^2 - K_{\alpha\beta}K^{\alpha\beta}) \\ +4\pounds_{\boldsymbol{v}}K) - Kh_{\rho\nu}(K^2 - K_{\alpha\beta}K^{\alpha\beta} + 2\pounds_{\boldsymbol{v}}K)]) = 0$$

(4,2) theory in the electric limit

The evolution equation is

$$\begin{aligned}
&2(KK_{\mu\nu} - K_{\mu}^{\sigma}K_{\nu\sigma})(1+\beta'(2(K^2 - K_{\alpha\beta}K^{\alpha\beta}) + 4\pounds_{\boldsymbol{v}}K)) \\
&+ 2(2\beta'(K^2 - K_{\alpha\beta}K^{\alpha\beta}) - \beta'\pounds_{\boldsymbol{v}}K)(\pounds_{\boldsymbol{v}}K_{\mu\nu} - 4K_{\mu}^{\sigma}K_{\sigma\nu}) \\
&+ \pounds_{\boldsymbol{v}}\left[(Kh_{\mu\nu} - K_{\mu\nu})(1+\beta'(2(K^2 - K_{\alpha\beta}K^{\alpha\beta}) + 4\pounds_{\boldsymbol{v}}K))\right] \\
&- 8\beta'\pounds_{\boldsymbol{v}}\left[K_{\mu\nu}(2(K^2 - K_{\alpha\beta}K^{\alpha\beta}) - \pounds_{\boldsymbol{v}}K)\right] \\
&+ 2\beta'\pounds_{\boldsymbol{v}}\pounds_{\boldsymbol{v}}\left[2(K^2 - K_{\alpha\beta}K^{\alpha\beta}) - \pounds_{\boldsymbol{v}}K\right] = 0.
\end{aligned}$$

- Call this system of equations "system 1". It is the system of equations for the whole electric theory.
- When $\beta' = 0$, it reduces to GR.

Special case

- When $\beta' \neq 0$, there exists a system of equations, call it system 2:

$$\begin{aligned} \pounds_{\boldsymbol{v}} K &= \tfrac{-2}{5\beta'}, \\ K^2 - K_{\mu\nu}K^{\mu\nu} &= \tfrac{-1}{5\beta'}, \\ -2K_\mu^\sigma K_{\sigma\nu} + Kh_{\mu\nu} &= \pounds_{\boldsymbol{v}} K_{\mu\nu}, \\ h^{\rho\sigma}\nabla_\sigma(Kh_{\rho\mu} - K_{\rho\mu}) &= 0. \end{aligned}$$

  whose solutions are a subset of the solution set of system 1.
- Any solution of system 2 is a solution of system 1 i.e. to the full electric theory.
- System 2 resembles GR with a cosmological constant equal to $\frac{-1}{10\beta'}$.

(4,2) theory in the magnetic limit

The magnetic action is

$$S = c^3 \int e\big[\overset{c}{R} + \beta'(-K^2 + K_{\mu\nu}K^{\mu\nu} + 2\pounds_{\boldsymbol{v}}K)(\overset{c}{R} + \nabla_\mu(v^\lambda b_\lambda{}^\mu))\big]\, d^4x,$$

where $b_{\mu\nu} = \partial_\mu \tau_\nu - \partial_\nu \tau_\mu$.

- Constraints and the evolution equation for the electric action must be imposed on the magnetic action.
- The theory should have a solution corresponding to the Carrollian limit of Schwarzchild-Bach black hole solutions with a cosmological constant.

Outlook

- (The Carrollian limit of) Schwarzchild-Bach black holes as solutions for (2,4) and (4,2).
- Dynamics of particles near such black holes' horizons.
- The equivalence between the PUL parameterization approach and the zero signature approach (Henneaux, 1979) for a generic gravity theory.

Thank you