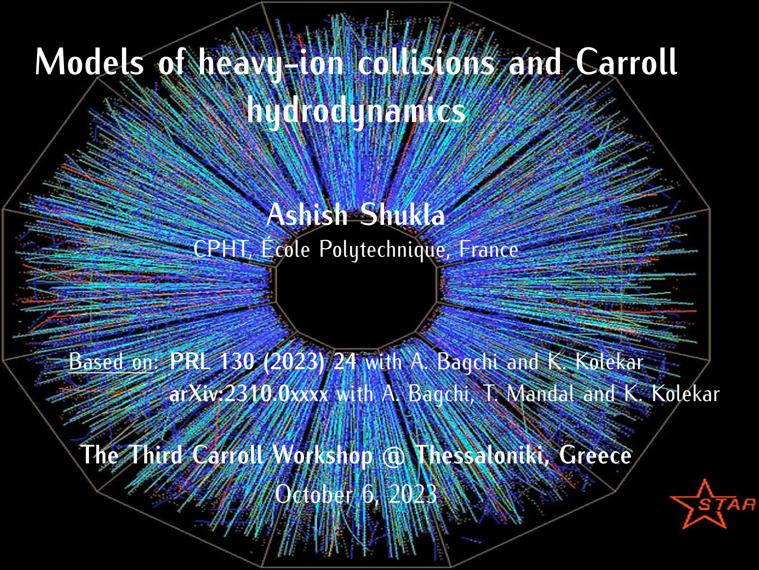

Models of heavy-ion collisions and Carroll hydrodynamics

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- In this talk, I would like to present a duality between certain analytic models of heavy-ion collisions and Carroll hydrodynamics, which arises in the $c \to 0$ limit of relativistic hydrodynamics.
- Ultrarelativistic heavy-ion collisions, such as between Pb-Pb or Au-Au ions, are used to create extreme energy densities and temperatures, that can "melt" the colliding nucleons into free quarks and gluons $\equiv$ QGP.
- Remarkably, the QGP behaves like an almost perfect fluid!

Specific viscosity $\sim 0.1 - 0.2$[1] : very close to the KSS bound $\eta/s = 1/4\pi \approx 0.08$.[2]

The state of our universe for the first few microseconds after its big bang birth.

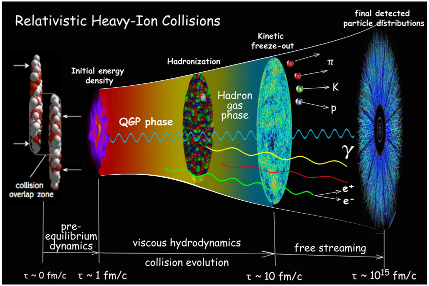

Figure from Shen and Heinz arXiv:1507.01558

[1] J. E. Parkkila et al, Phys. Rev. C 104, 054904

[2] P. Kovtun, D. T. Son & A. O. Starinets, Phys. Rev. Lett. 94 (2005) 111601

- The spacetime evolution of the QGP has some symmetries:
  a. Approximate boost invariance
  b. Approximate rotation invariance along the beam axis
- Using these symmetries (and some more), one can construct analytic models for the hydrodynamics of the QGP.
- Bjorken flow: boost invariance + translation invariance in the transverse plane.[1]
- Gubser flow: boost invariance + rotation invariance + conformal invariance in the transverse plane.[2,3]
- These phenomenological assumptions fix the four-velocity profile of the fluid, which can then be used in the hydrodynamic equations for the flow.
- Duality with Carroll hydrodynamics maps these assumptions into geometric properties of the manifold on which the Carroll fluid lives, yielding the same hydrodynamics equations!
- Dynamics of QGP: a new entry into the Carrollian kaleidoscope.

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[1] J. D. Bjorken, *Phys. Rev. D 27, 140 (1983)*

[2] S. S. Gubser, *Phys. Rev. D 82, 085027 (2010)*

[3] S. S. Gubser and A. Yarom, *Nucl. Phys. B 846, 469 (2011)*

Outline of the talk

MOTIVATION ✓

MODELS OF QGP EVOLUTION
- Bjorken Flow
- Gubser Flow

CARROLL HYDRODYNAMICS

DUALITY MAP

VISCOUS EFFECTS

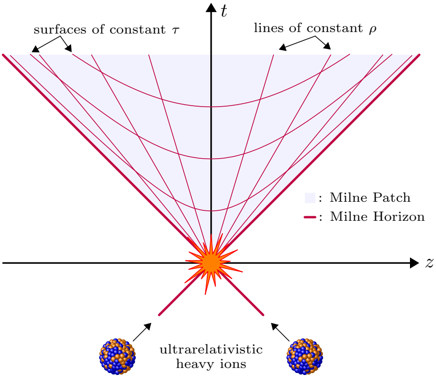
DISCUSSION & OUTLOOK

Bjorken Flow

- The setup: choose z to be the beam axis, and x, y as the transverse plane.
- Simplify: assume the collision happens at $t = 0$ and $z = 0$.
- Work in Milne coordinates:

$$\tau = \sqrt{t^2 - z^2}\,, \quad \rho = \frac{1}{2}\log\left(\frac{t+z}{t-z}\right)\,, \quad x\,, \quad y$$

- Minkowski metric: $ds^2 = -d\tau^2 + \tau^2 d\rho^2 + dx^2 + dy^2$



- Demanding invariance of the fluid flow u^μ under a spacetime transformation generated by $\xi \equiv \xi^\mu \partial_\mu$ means $\pounds_\xi u^\mu = 0$.
- Bjorken flow: u^μ boost inv. + translation & rotation inv. in transverse plane $\Rightarrow u^\mu = (1, 0, 0, 0)$
- Perfect fluid energy-momentum tensor: $T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu}$.
- Hydrodynamic equations: $\nabla_\mu T^{\mu\nu} = 0$.
- With the Bjorken flow profile, hydro equation becomes:

$$\frac{d\epsilon}{d\tau} = -\frac{\epsilon + P}{\tau}$$

- Given an equation of state $P = P(\epsilon)$, one can compute the evolution of the energy density of the QGP.

Gubser Flow

- At the extreme energies of the QGP, the dynamics is approximately conformal.
- Symmetries: boost inv. + rotation inv. about beam axis + conf. inv. under

$$\xi_1 = \partial_x + q^2 \left(2 x x^\mu \partial_\mu - x^\mu x_\mu \partial_x\right),$$
$$\xi_2 = \partial_y + q^2 \left(2 y x^\mu \partial_\mu - x^\mu x_\mu \partial_y\right).$$

  Here q is a tunable parameter with dim $= L^{-1}$.
- Along with the generator of rotations $\xi_{\text{rot}} = x\partial_y - y\partial_x$, ξ_1, ξ_2 form an $SO(3)_q$ subgroup of the full conformal group $SO(4,2)$.

$$[\xi_1, \xi_2] = -4q^2 \xi_{\text{rot}},$$
$$[\xi_1, \xi_{\text{rot}}] = \xi_2,$$
$$[\xi_2, \xi_{\text{rot}}] = -\xi_1.$$

  The $SO(3)_q$ group above commutes with the $SO(1,1)$ subgroup of $SO(4,2)$ corresponding to boosts along the z-axis, generated by $\xi_{\text{boost}} = z\partial_t + t\partial_z$.

- Conf. inv. of the flow:

$$\pounds_{\xi_1} u^\mu = -\frac{1}{4}\left(\nabla_\nu \xi_1^\nu\right) u^\mu, \quad \pounds_{\xi_2} u^\mu = -\frac{1}{4}\left(\nabla_\nu \xi_2^\nu\right) u^\mu.$$

- Note that the generators ξ_a, $a = 1, 2$ are conformal isometries for the background metric, justifying the requirement above

$$\pounds_{\xi_a} g_{\mu\nu} = \frac{1}{2}\left(\nabla_\alpha \xi_a^\alpha\right) g_{\mu\nu}.$$

- With these symmetry assumptions the fluid velocity becomes

$$u^\mu = (\cosh\kappa(\tau, r), 0, \sinh\kappa(\tau, r), 0)$$
$$\kappa(\tau, r) = \tanh^{-1}\left[\frac{2q^2\tau r}{1 + q^2(\tau^2 + r^2)}\right].$$

  Interestingly, the flow now has a nontrivial radial dependence!

- Use the conformal eqn. of state: $P = \epsilon/3$.

- The hydro equations $\nabla_\mu T^{\mu\nu} = 0$ are now given by

$$\frac{\partial \epsilon}{\partial \tau} = \frac{4\epsilon}{3}\left(\frac{\cosh 2\kappa - 2}{\tau} - \frac{\sinh 2\kappa}{r}\right)$$
$$\frac{\partial \epsilon}{\partial r} = \frac{4\epsilon}{3}\left(\frac{\cosh 2\kappa - 1}{r} - \frac{\sinh 2\kappa}{\tau}\right)$$

- They admit the solution

$$\epsilon(\tau, r) = \frac{\epsilon_0}{\tau^{4/3}} \frac{(2q)^{8/3}}{[1 + 2q^2(\tau^2 + r^2) + q^4(\tau^2 - r^2)^2]^{4/3}}.$$

As with the velocity profile, we now have an interesting dependence on the radial coordinate r in the energy density as well.

Gubser flow on $dS_3 \times \mathbb{R}$ background

- One can recast Gubser flow on a $dS_3 \times \mathbb{R}$ background as well.
- Perform a Weyl rescaling $ds^2 \to ds^2/\tau^2$ on the flat metric in Milne coordinates, followed by the transformation $(\tau, r) \to (\varsigma, \psi)$, where

$$\sinh \varsigma = -\frac{1 - q^2(\tau^2 - r^2)}{2q\tau}, \quad \tan \psi = \frac{2qr}{1 + q^2(\tau^2 - r^2)}.$$

- The metric becomes $ds^2 = -d\varsigma^2 + \cosh^2 \varsigma \left(d\psi^2 + \sin^2 \psi \, d\phi^2 \right) + d\rho^2$.
- This is the metric on $dS_3 \times \mathbb{R}$, where (ς, ψ, ϕ) are coordinates on the three-dimensional global de Sitter spacetime.
- The $SO(3)_q$ conformal symmetry of Milne is now an exact isometry, associated with the spherical symmetry of the S^2 parametrized by ψ, ϕ.
- Demanding invariance under boosts and rotation gives $u^\mu = (1, 0, 0, 0)$.
- The hydro equation for Gubser flow now becomes

$$\partial_\varsigma \epsilon = -\tfrac{8\epsilon}{3} \tanh \varsigma$$

This has the solution: $\epsilon(\varsigma) = \epsilon_0 / (\cosh \varsigma)^{8/3}$.

Carroll Hydrodynamics

- Definition: the $c \to 0$ limit of relativistic hydrodynamics.
- The Papapetrou-Randers parametrization makes the $c \to 0$ limit quite transparent.[1,2]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -c^2(\Omega dt - b_i dx^i)^2 + a_{ij} dx^i dx^j ,$$
$$g^{00} = \frac{-1 + c^2 b^2}{\Omega^2} , \quad g^{0i} = g^{i0} = \frac{c}{\Omega} b^i , \quad g^{ij} = a^{ij} .$$

  Ω, b_i, a_{ij} are functions of $(t, \mathbf{x})$.
  $b^2 \equiv b^i b_i$, where the index on b_i can be raised using a^{ij}.

- The $c \to 0$ limit gives a Carroll manifold with a degenerate metric:

$$d\ell^2 = h_{\mu\nu} dx^\mu dx^\nu = a_{ij}(t, \mathbf{x}) dx^i dx^j , \quad v = \frac{1}{\Omega(t, \mathbf{x})} \partial_t , \quad h_{\mu\nu} v^\nu = 0.$$

- Define Carroll covariant temporal and spatial derivatives:

$$\hat{\partial}_t \equiv \frac{1}{\Omega} \partial_t , \quad \hat{\partial}_i \equiv \partial_i + \frac{b_i}{\Omega} \partial_t .$$

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[1] L. Ciambelli et al, *JHEP 1807 (2018) 165* and *Class. Quantum Grav. 35 (2018) 165001*
[2] A. C. Petkou et al, *JHEP 2209 (2022) 162*

- Consider now a perfect fluid on this background, with EM tensor

$$T^{\mu\nu} = (\epsilon + P)\frac{u^\mu u^\nu}{c^2} + P g^{\mu\nu}.$$

- A convenient parametrization for the fluid velocity is $u = \gamma\partial_t + \gamma v^i \partial_i$, where

$$\gamma = \frac{1 + c^2 \vec{b}\cdot\vec{\beta}}{\Omega\sqrt{1 - c^2\beta^2}}\,, \quad v^i = \frac{c^2\Omega\beta^i}{1 + c^2\vec{b}\cdot\vec{\beta}}\,.$$

Here $\beta^i(t, \mathbf{x})$ is a Carrollian vector field.

- The next step is to take the $c \to 0$ limit. One has to assume sensible scaling behaviour for thermodynamic quantities in this limit.

$$\epsilon = \varepsilon + \mathcal{O}(c^2)\,, \quad P = p + \mathcal{O}(c^2).$$

Also,

$$u^0 = \frac{c}{\Omega} + \mathcal{O}(c^3)\,, \qquad u^i = c^2\beta^i + \mathcal{O}(c^4)\,,$$
$$u_0 = -c\,\Omega + \mathcal{O}(c^3)\,, \quad u_i = c^2(b_i + \beta_i) + \mathcal{O}(c^4)\,.$$

- These lead to the EM tensor behaviour:

$$T^0{}_0 = -\varepsilon + \mathcal{O}(c^2)\,, \quad T^i{}_j = p\,\delta^i{}_j + \mathcal{O}(c^2)\,,$$
$$T^0{}_i = \frac{c}{\Omega}(\varepsilon + p)(b_i + \beta_i) + \mathcal{O}(c^3)\,, \quad T^i{}_0 = -c\,\Omega(\varepsilon + p)\beta^i + \mathcal{O}(c^3)\,.$$

- Finally, putting all this together leads to the equations of Carroll hydrodynamics for a perfect fluid:

$$\hat{\partial}_t \varepsilon = -\theta(\varepsilon + p),$$
$$\hat{\partial}_i p = -\varphi_i(\varepsilon + p) - \left(\hat{\partial}_t + \theta\right)\left[(\varepsilon + p)\beta_i\right].$$

Here θ is Carrollian expansion and φ_i is Carrollian acceleration, given via

$$\theta \equiv \frac{1}{\Omega}\partial_t \log\sqrt{a}, \quad \varphi_i \equiv \frac{1}{\Omega}(\partial_t b_i + \partial_i \Omega),$$

where $a = \det a_{ij}$.

- For a conformal Carroll fluid, one sets $\varepsilon = 3p$.

Duality Maps

Bjorken flow

- Suppose we now specialize to the following geometric data on the Carroll manifold:

$$\Omega = 1\,, \quad b_i = -\beta_i\,, \quad a_{ij}dx^i dx^j = \tau^2 d\rho^2 + dx^2 + dy^2.$$

- The equations of Carroll hydrodynamics then become:

$$\partial_\tau \varepsilon = -\frac{\varepsilon + p}{\tau}\,, \quad \partial_i p = 0.$$

- The first equation is the equation for Bjorken flow, while the second implies boost-invariance (using e.o.s. $p = p(\varepsilon)$).
- Thus the geometric data above pertains to a Carroll fluid which has the same dynamics as Bjorken flow.
  The key point is that the phenomenological assumptions put into Bjorken flow are now captured by the geometry of the Carroll manifold!

Gubser flow

- Specialize to the Carroll manifold with the geometric properties

$$\Omega = \cosh\kappa,$$
$$b_r = -\beta_r + \sinh\kappa, \quad b_\rho = -\beta_\rho, \quad b_\phi = -\beta_\phi,$$
$$a_{ij}dx^i dx^j = \tau^2 d\rho^2 + \left[1 + 2q^2(\tau^2 + r^2) + q^4(\tau^2 - r^2)^2\right](dr^2 + r^2 d\phi^2).$$

- The Carroll hydro equations now become

$$\partial_\tau \varepsilon = \frac{4\varepsilon}{3}\left(\frac{\cosh 2\kappa - 2}{\tau} - \frac{\sinh 2\kappa}{r}\right),$$
$$\partial_r \varepsilon = \frac{4\varepsilon}{3}\left(\frac{\cosh 2\kappa - 1}{r} - \frac{\sinh 2\kappa}{\tau}\right),$$
$$\partial_\rho \varepsilon = 0, \quad \partial_\phi \varepsilon = 0.$$

These are nothing but the equations for Gubser flow, along with the symmetry assumptions!

- The Carroll manifold above thus geometrizes the assumptions that go into Gubser's model for the QGP, and provides a dual description in terms of Carroll hydrodynamics.

- The duality is not limited to the choice of Milne coordinates.
- For instance, for Gubser flow on the $dS_3 \times \mathbb{R}$ background, the dual Carroll description is provided by a Carroll fluid on a Carrollian manifold with the geometric properties:

$$\Omega = 1, \quad b_i = -\beta_i, \quad a_{ij} dx^i dx^j = \cosh^2 \varsigma \left(d\psi^2 + \sin^2 \psi \, d\phi^2 \right) + d\rho^2.$$

- In this case, the Carroll fluid equations become

$$\partial_\varsigma \varepsilon = -\frac{8\varepsilon}{3} \tanh \varsigma, \quad \partial_i \varepsilon = 0.$$

These are indeed the equations for Gubser flow on the $dS_3 \times \mathbb{R}$ background, complete with the symmetry assumptions.

Including Viscous Corrections

- We can now depart from the approximation of perfect fluids and include hydrodynamic derivative corrections.
- At the first order in derivatives, the EM tensor is

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} - \eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\nabla \cdot u$$

Here $\sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\frac{\nabla_\alpha u_\beta + \nabla_\beta u_\alpha}{2}\right) - \frac{1}{3}\Delta^{\mu\nu}\nabla \cdot u, \quad \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$. Also, $\eta, \zeta \geq 0$ are the shear and bulk viscosities that lead to dissipative effects.

- For Bjorken flow, the hydro equation including viscous effects is

$$\frac{d\epsilon}{d\tau} = -\frac{\epsilon + P}{\tau} + \frac{1}{\tau^2}\left(\frac{2}{3}\eta + \zeta\right)$$

- For Gubser flow ($\zeta = 0, \eta = \eta_o\epsilon^{3/4}$), the viscous hydro equations are

$$\partial_\tau\epsilon = \frac{4\epsilon}{3}\left(\frac{\cosh 2\kappa - 2}{\tau} - \frac{\sinh 2\kappa}{r}\right) + \frac{2\eta_o\epsilon^{3/4}}{3\,\mathrm{sech}^3\kappa}\left(\frac{1}{\tau} - \frac{\tanh\kappa}{r}\right)^2,$$

$$\partial_r\epsilon = \frac{4\epsilon}{3}\left(\frac{\cosh 2\kappa - 1}{r} - \frac{\sinh 2\kappa}{\tau}\right) - \frac{2\eta_o\epsilon^{3/4}\sinh\kappa}{3\,\mathrm{sech}^2\kappa}\left(\frac{1}{\tau} - \frac{\tanh\kappa}{r}\right)^2.$$

- Let us now incorporate viscous corrections on the Carroll side.
- The EM tensor is $T^{\mu\nu} = (\epsilon + P)\frac{u^\mu u^\nu}{c^2} + Pg^{\mu\nu} - \eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\nabla \cdot u$.
- Taking the $c \to 0$ limit of $\nabla_\mu T^{\mu\nu} = 0$ now gives the viscous Carroll hydrodynamic equations

$$\hat{\partial}_t \varepsilon = -\theta(\varepsilon + p - \tilde{\zeta}\theta) + \tilde{\eta}\xi^{ij}\xi_{ij},$$
$$\hat{\partial}_i p = -\varphi_i(\varepsilon + p) + \left(\hat{\nabla}_j + \varphi_j\right)\left[\tilde{\eta}\xi^j{}_i + \tilde{\zeta}\theta\delta^j{}_i\right]$$
$$- \left(\hat{\partial}_t + \theta\right)\left[(\varepsilon + p)\beta_i - \beta_j(\tilde{\eta}\xi^j{}_i + \tilde{\zeta}\theta\delta^j{}_i)\right].$$

Here we have made use of the scaling assumptions

$$\eta = \tilde{\eta} + \mathcal{O}(c^2), \quad \zeta = \tilde{\zeta} + \mathcal{O}(c^2).$$

- It is straight forward to check that the equations above reduce to the equations of viscous Bjorken/Gubser flow, employing the duality maps proposed earlier.
- Thus the duality is not limited to perfect fluids only.

Discussion and Outlook

- Main message: Duality between Carroll fluids on manifolds with specific geometric data, and models of QGP evolution.
- Exploit the duality from the Carroll side – compute the subleading corrections in the $c \to 0$ limit.
  On the QGP side, they would imply how to systematically depart from the phenomenological assumptions for Bjorken/Gubser flow.

Many more directions to explore...

- Connections with black hole membrane paradigm,[1,2] where the equations for the fluid are Carrollian.[3]
- Connections with flat holography?
- ...
- ...

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[1] T. Damour, *Phys. Rev. D 18, 3598 (1978)*

[2] R. H. Price and K. S. Thorne, *Phys. Rev. D 33, 915 (1986)*

[3] L. Donnay and C. Marteau, *Class. Quant. Grav. 36, 165002 (2019)*

Thank you for your attention!